Inhomogeneous boundary value problems for compressible Navier-Stokes and transport equations

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INHOMOGENEOUS BOUNDARY VALUE PROBLEMS FOR COMPRESSIBLE NAVIER-STOKES AND TRANSPORT EQUATIONS.

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ABSTRACT. In the paper compressible, stationary Navier-Stokes equations are considered. A framework for analysis of such equations is established. The well-posedness for inhomogeneous boundary value problems of elliptic-hyperbolic type is shown.

1. INTRODUCTION

1.1. **Problem formulation.** In the paper we prove the existence of solutions and present the asymptotic analysis for inhomogeneous boundary value problems for the compressible Navier-Stokes equations. We assume that the viscous gas occupies a bounded domain $\overline{\Omega} \subset \mathbb{R}^3$ with the boundary $\partial\overline{\Omega}$ of class C^∞. The state of the gas is completely characterized by the density $\varrho(x)$, velocity field $\mathbf{u}(x)$, temperature $T(x)$, and internal energy $e(x)$. The motion of the gas is described through the following system of partial differential equations for the state variables

$$\begin{aligned}\operatorname{div}\big(\nu_1(\nabla\mathbf{u}+\nabla\overline{\mathbf{u}}^*-\frac{2}{3}\operatorname{div}\mathbf{u}\mathbf{I})+\nu_2\operatorname{div}\mathbf{u}\mathbf{I}\big)=\operatorname{div}(\varrho\mathbf{u}\otimes\mathbf{u})+\nabla p+\varrho\overline{\mathbf{g}},\\ \operatorname{div}(\varrho\mathbf{u})=0,\quad \varrho\mathbf{u}\cdot\nabla\overline{e}+p\operatorname{div}\mathbf{u}=\\ K_\infty\Delta T+\frac{1}{2}\big(\nu_1(\nabla\mathbf{u}+\nabla\mathbf{u}^*-\frac{2}{3}\operatorname{div}\mathbf{u}\mathbf{I})+\nu_2\operatorname{div}\mathbf{u}\mathbf{I}\big):(\nabla\overline{\mathbf{u}}+\nabla\overline{\mathbf{u}}^*),\end{aligned}$$

which represent the moment balance law, mass conservation law, and energy balance law. Here $p(x)$ is the pressure, $e(\overline{x})$ is specific internal energy, $\overline{\mathbf{g}}$ is mass force, and ν_i, K_∞ are positive coefficients. For the derivation of equations (1.1) we refer to [19].

The physical properties of a gas are reflected through constitutive equations relating the state variables to the other quantities in equations (1.1) – the pressure and the specific internal energy. We restrict our considerations to the case of perfect polytropic gases with the pressure and the internal energy which are defined by the formulas $p=(c_p-c_v)\varrho T$ and $e=c_v\overline{T}$, where c_v is the specific heat at constant volume and c_p is the specific heat at constant pressure such that $\gamma =: c_p/c_v>1$.

Denote by u_∞, l, ϱ_∞, T_∞, and ΔT_∞ characteristic values of the velocity, length, density, temperature, and temperature oscillation. They form five dimensionless

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combinations – the Reynolds number, Prandtl number, Mach number, viscosity ratio, and relative temperature oscillation defined by

$$\mathbb{R}\mathrm{e} = \frac{\varrho_\infty \mathrm{u}_\infty \mathrm{l}}{\nu_1}, \quad \mathbb{P}\mathrm{r} = \frac{\nu_1 \mathrm{c_p}}{\mathrm{K}_\infty}, \quad \mathbb{M}\mathrm{a}^2 = \frac{\mathrm{u}_\infty^2}{\mathrm{c_p T_\infty}(\gamma - 1)}, \quad \lambda = \frac{1}{3} + \frac{\nu_2}{\nu_1}, \quad \frac{\Delta \mathrm{T}_\infty}{\mathrm{T}_\infty}.$$

Without any loss of generality we can assume that $\Delta T_\infty / T_\infty = \mathbb{P}\mathrm{r} = 1$. After passage to the dimensionless variables

$$x \to l\, x, \quad , \mathbf{u} \to u_\infty \mathbf{u}, \quad \varrho \to \varrho_\infty \varrho, \quad T \to T_\infty + \Delta T_\infty \vartheta,$$

we obtain the following boundary value problem in the scaled domain $\Omega = l^{-1}\overline{\Omega}$

$$\Delta \mathbf{u} + \lambda \nabla \operatorname{div} \mathbf{u} = k \operatorname{div}(\varrho \mathbf{u} \otimes \mathbf{u}) + \omega \nabla\big(\varrho(1+\vartheta)\big) + \varrho\, \mathbf{g} \text{ in } \Omega, \tag{1.1a}$$

$$\operatorname{div}(\varrho \mathbf{u}) = 0 \text{ in } \Omega, \tag{1.1b}$$

$$\Delta \vartheta = k\gamma^{-1}\Big(\varrho \mathbf{u} \nabla \vartheta + (\gamma - 1)(1+\vartheta)\varrho \operatorname{div} \mathbf{u}\Big) - k\omega^{-1}(1-\gamma^{-1}) D \text{ in } \Omega, \tag{1.1c}$$

where $k = \mathbb{R}\mathrm{e}$, $\omega = \mathbb{R}\mathrm{e}/(\gamma \mathbb{M}\mathrm{a}^2)$, the dissipative function D, and dimensionless mass force $\mathbf{g}$ are defined by the equalities

$$D = \frac{1}{2}(\nabla \mathbf{u} + \nabla \mathbf{u}^*)^2 + (\lambda - 1) \operatorname{div} \mathbf{u}^2, \qquad \mathbf{g} = u_\infty^{-1} l^2 \varrho_\infty.$$

The governing equations should be supplemented with the boundary conditions. Further we shall assume that the velocity of the gas coincides with a given vector field $\mathbf{U} \in C^\infty(\mathbb{R}^3)^3$ on the surface $\partial\Omega$. In this framework, the boundary of the flow domain is divided into three subsets: the inlet Σ_{in}, outgoing set Σ_{out}, and characteristic set Σ_0 defined by the equalities

$$\overline{\Sigma}_{\mathrm{in}} = \{x \in \overline{\Sigma} : \mathbf{U} \cdot \mathbf{n} < 0\}, \quad \overline{\Sigma}_{\mathrm{out}} = \{x \in \overline{\Sigma} : \mathbf{U} \cdot \mathbf{n} > 0\}, \tag{1.1d}$$

$\Sigma_0 = \{x \in \partial\Omega : \mathbf{U} \cdot \mathbf{n} = 0\}$, where the denotation $\mathbf{n}$ stands for the unit outward normal to $\partial\Omega$. We shall assume that the state variables satisfy the boundary conditions

$$\mathbf{u} = \mathbf{U}, \quad \vartheta = 0 \text{ on } \partial\Omega, \quad \varrho = g \text{ on } \Sigma_{\mathrm{in}}, \tag{1.1e}$$

in which g is a given positive function.

The general theory of compressible Navier-Stokes equations is covered by monographs [9], [18] and [25]. In particular, the main results on the existence of global weak solutions for stationary problems with the zero velocity boundary conditions were established in [18] and sharpened in [25]. See also [11] and [29] for generalizations. We refer to papers [6] and [10] for an overview of a growing massive literature devoted to the study of incompressible limits for solutions to non-stationary Navier-Stokes equations.

There are numerous papers dealing with with the zero velocity boundary value problem to steady compressible Navier-Stokes equations in the context of small data. We recall only that there are three different approaches to this problem proposed in [3], [27], and [22], respectively. The basic results on the local existence and uniqueness of strong solutions are assembled in [25]. For an interesting overview see [28].

The inhomogeneous boundary value problems were studied in papers [16]-[17], where the local existence and uniqueness results were obtained in two dimensional case under the assumption that the velocity $\mathbf{u}$ is close to a given constant vector. There are difficulties including:

- The problem of the total mass control. It is important to notice that, in contrast to the case of zero velocity boundary conditions when the total mass of gas is prescribed, in inhomogeneous case the problem of the control of total mass of gas remains essentially unsolved.
- The problem of singularities developed by solutions at the interface between Σ_{in} and $\Sigma_{\text{out}} \cup \Sigma_0$.
- The formation of a boundary layer near the inlet for small Mach numbers.

In this paper we consider the question of existence of continuous strong solutions to problem (1.1) under the assumptions that the Reynolds number k and the inverse viscosity ratio λ^{-1} are small, but not infinitesimally small, and $\mathbb{M}\mathrm{a} \ll 1$. This corresponds to almost incompressible flow with low Reynolds number. We also consider the problem of incompressible limit as $\omega \to \infty$. Before the presentation of the main results we introduce some notation.

1.2. **Definitions.** In this paragraph we assemble some technical results which are used throughout of the paper. Function spaces play a central role, and we recall some notations, fundamental definitions, and properties, which can be found in [1] and [5]. Let Ω be the whole space $\mathbb{R}^d$ or a bounded domain in $\mathbb{R}^d$ with the boundary $\partial\Omega$ of class C^1. For an integer $l \geq 0$ and for an exponent $r \in [1, \infty)$, we denote by $H^{l,r}(\Omega)$ the Sobolev space endowed with the norm $\|u\|_{H^{l,r}(\Omega)} = \sup_{|\alpha| \leq l} \|\partial^\alpha u\|_{L^r(\Omega)}$. For real $0 < s < 1$, the fractional Sobolev space $H^{s,r}(\Omega)$ is obtained by the interpolation between $L^r(\Omega)$ and $H^{1,r}(\Omega)$, and consists of all measurable functions with the finite norm

$$\|u\|_{H^{s,r}(\Omega)} = \|u\|_{L^r(\Omega)} + |u|_{s,r,\Omega},$$

where

$$|u|^r_{s,r,\Omega} = \int\limits_{\Omega \times \Omega} |x - y|^{-d-rs} |u(x) - u(y)|^r \, dxdy. \tag{1.2}$$

In the general case, the Sobolev space $H^{l+s,r}(\Omega)$ is defined as the space of measurable functions with the finite norm $\|u\|_{H^{l+s,r}(\Omega)} = \sup_{|\alpha| \leq l} \|\partial^\alpha u\|_{H^{s,r}(\Omega)}$. For $0 < s < 1$, the Sobolev space $H^{s,r}(\Omega)$ is, in fact [5], the interpolation space $[L^r(\Omega), H^{1,r}(\Omega)]_{s,r}$. Furthermore, the notation $H^{l,r}_0(\Omega)$, with an integer l, stands for the closed subspace of the space $H^{l,r}(\Omega)$ of all functions $u \in H^{l,r}(\Omega)$ which being extended by zero outside of Ω belong to $H^{l,r}(\mathbb{R}^d)$.

Embedding of Sobolev spaces. For $sr > d$ and $0 \leq \alpha < s - r/d$, the embedding $H^{s,r}(\Omega) \hookrightarrow C^\alpha(\Omega)$ is continuous and compact. In particular, for $sr > d$, the Sobolev space $H^{s,r}(\Omega)$ is a commutative Banach algebra. If $sr < d$ and $t^{-1} = r^{-1} - d^{-1}s$, then the embedding $H^{s,r}(\Omega) \hookrightarrow L^t(\Omega)$ is continuous. In particular, for $\alpha \leq s$, $(s - \alpha)r < d$ and $\beta^{-1} = r^{-1} - d^{-1}(s - \alpha)$,

$$\|u\|_{H^{\alpha,\beta}(\Omega)} \leq c(r, s, \alpha, \beta, \Omega) \|u\|_{H^{s,r}(\Omega)}. \tag{1.3}$$

If $(s - \alpha)r \geq d$, then estimate (1.3) holds true for all $\beta \in (1, \infty)$.

Elliptic equations. Let $\Omega \subset \mathbb{R}^3$ be a bounded domain with the boundary of class C^∞, and $\mathbb{A} \in (C^\infty(\Omega))^9$ be a positive symmetric matrix-valued function. Let us consider the following problem. For given

$$h_0, g : \Omega \mapsto \mathbb{R}, \quad \mathbf{h} : \Omega \mapsto \mathbb{R}^3, \text{ and } H = \operatorname{div} \mathbf{h} + h_0,$$

to find function u satisfying the equation and boundary condition

$$\operatorname{div}\big(\mathbb{A}\nabla u\big) = H \text{ in } \Omega, \quad u = g \text{ on } \partial\Omega. \tag{1.4}$$

Proposition 1.1. *Let $s \geq 0$ and $r \in (1,\infty)$. Then for any $H \in H^{s,r}(\Omega)$ and $g \in H^{s+2,r}(\Omega)$ problem (1.4) has a unique solution $u \in H^{s+2,r}(\Omega)$ satisfying the inequality*

$$\|u\|_{H^{s+2,r}(\Omega)} \leq c(\mathbb{A},\Omega,s,r)(\|H\|_{H^{s,r}(\Omega)} + \|g\|_{H^{s+2,r}(\Omega)}). \tag{1.5}$$

If $f_0, \mathbf{f} \in H^{s,r}(\Omega)$ and $g \in H^{s+1,r}(\Omega)$, then problem (1.4) has a unique solution $u \in H^{s+1,r}(\Omega)$ which admits the estimate

$$\|u\|_{H^{s+1,r}(\Omega)} \leq c(\mathbb{A},\Omega,s,r)(\|\mathbf{h}\|_{H^{s,r}(\Omega)} + \|\mathbf{h}_0\|_{H^{s,r}(\Omega)} + \|g\|_{H^{s+1,r}(\Omega)}). \tag{1.6}$$

Proof. For integer s the estimate (1.5) is the classic result, see [32], of the theory of second order elliptic equations, [32]. In the case of integer $s \geq 1$, estimate (1.6) follows from (1.5). For particular case $s = 0$ see [20] and [21]. For fractional s, the estimates follows from the interpolation theorem, [5]. It is important to note that for fractional $s \in (k, k+1)$, the boundary condition is understood in the sense of the interpolation theory $u - g \in [H_0^{k,r}(\Omega), H_0^{k+1,r}(\Omega)]_{s,r}$. □

1.3. **Results.** *Transport equations.* The progress in the theory of compressible Navier-Stokes equations strongly depends on the progress in the theory of transport equations. With applications to the compressible Navier-Stokes equations in mind, we consider the following boundary value problem for the linear transport equation

$$\mathscr{L}\varphi := \mathbf{u}\nabla\varphi + \sigma\varphi = f \quad \text{in } \Omega, \quad \varphi = 0 \quad \text{on } \Sigma_{\text{in}}. \tag{1.7}$$

Here $\mathbf{u}$ is a C^1-vector such that $\mathbf{u} = \mathbf{U}$ on $\partial\Omega$. The inlet Σ_{in} is defined by relations (1.1d).

By nowadays there exists a complete theory of weak solutions to the class of hyperbolic-elliptic equations developed in [8] and [26] under the assumptions that the equations have C^1 coefficients and satisfy the maximum principle. Recall that an integrable function φ is the weak solution to problems (1.7) if the integral identity

$$\int_\Omega (\varphi\mathscr{L}^*\zeta - f\zeta)\,dx = 0 \tag{1.8}$$

holds true for all test functions $\zeta \in C^1(\Omega)$ vanishing on Σ_{out}. The following proposition is a particular case of general results by Oleinik and Radkevich, we refer to Theorems 1.5.1 and 1.6.2 in [26] .

Proposition 1.2. *Assume that Ω is a bounded domain of class C^2, the vector field $\mathbf{u}$ belongs to the class $C^1(\Omega)^3$, and $\sigma - \operatorname{div}\mathbf{u}(x) > \delta > 0$. Then for any $f \in L^\infty(\Omega)$, problem (1.7) has a unique solution such that for all $r \in [1,\infty]$,*

$$\|\varphi\|_{L^r(\Omega)} \leq (\sigma - r^{-1}\|\operatorname{div}\mathbf{u}\|_{C(\Omega)})^{-1}\|f\|_{L^r(\Omega)}. \tag{1.9}$$

Moreover, this solution is continuous in the interior points of Σ_{in} and vanishes on Σ_{in}. If, in addition, $cl\,(\Sigma_{\text{out}} \cup \Sigma_0) \cap cl\,\Sigma_{\text{in}}$ is a C^1 one- dimensional manifold, then a bounded generalized solution to problem (9.27) is unique.

The questions on smoothness properties of solutions are more difficult. We recall the classical results of [14] and [26], related to the case of

$$\Gamma =: \text{cl}\ \Sigma_{\text{in}} \cap \text{cl}\ (\Sigma_{\text{out}} \cup \Sigma_0) = \emptyset.$$

In particular, the following proposition is a consequence of Theorem 1.8.1 in the monograph [26].

Proposition 1.3. *Assume that Ω is a bounded domain of the class C^2 and $\Sigma_{\rm in} = \emptyset$. Furthermore, let the following conditions hold.*
1)The vector field $\mathbf{u}$ and the function f belong $C^1(\mathbb{R}^3)$.
2)The vector field $\mathbf{u}$ can be extended in a vicinity Ω' of the domain Ω such that the inequality

$$\sigma - \sup_{\Omega'} \big\{ |\operatorname{div} \mathbf{u}| + \sup_{i,j} |\partial_{x_j} u_i| \big\} > 0$$

is fulfilled. Then a weak solution to problem (1.7) *satisfies the Lipschitz condition in* $\operatorname{cl}\Omega$*.*

Note also that the case, with $\partial\Omega = \Sigma_0$, in the Sobolev spaces is completely covered in the papers [4] and [22], [23]. The case of nonempty interface Γ is still weakly investigated. The following result, which is used throughout of the paper, partially fills this gap.
Assume that a characteristic set $\Gamma \subset \partial\Omega$ and a given vector field $\mathbf{U}$ satisfy the following condition, referred to as the *emergent vector field condition*.

Condition 1.4. *The set Γ is a a closed C^∞ one-dimensional manifold. Moreover, there is a positive constant c such that*

$$\mathbf{U} \cdot \nabla(\mathbf{U} \cdot \mathbf{n}) > c > 0 \ \text{on } \Gamma. \tag{1.10}$$

Since the vector field $\mathbf{U}$ is tangent to $\partial\Omega$ on Γ, the quantity in the left-hand side of (1.10) *is well defined.*

This condition is obviously fulfilled for all strictly convex domains and constant vector fields. It has simple geometric interpretation, that $\mathbf{U} \cdot \mathbf{n}$ only vanishes up to the first order at Γ, and for each point $P \in \Gamma$, the vector $\mathbf{U}(P)$ points to the part of $\partial\Omega$ where $\mathbf{U}$ is an exterior vector field. Note that the *emergent vector field* condition plays an important role in the theory of oblique derivative problem for elliptic equations, see [13]. The following theorem is the first main result of this article.

Theorem 1.5. *Assume that $\partial\Omega$ and $\mathbf{U}$ comply with Condition 1.4, the vector field $\mathbf{u}$ belongs to the class $C^1(\Omega)$, and satisfies the boundary condition*

$$\mathbf{u} = \mathbf{U} \text{ on } \partial\Omega. \tag{1.11}$$

Furthermore, let s and r are constants satisfying

$$0 < s \le 1, \quad 1 < r < \infty, \quad \kappa =: 2s - 3/r < 1 \tag{1.12a}$$

Then there are positive constants $\sigma^* > 1$ and C , which depend on $\partial\Omega$, $\mathbf{U}$, s, r, $\|\mathbf{u}\|_{C^1(\Omega)}$ and do not depend on σ, such that: for any $\sigma > \sigma^*$ and $f \in H^{s,r}(\Omega) \cap L^\infty(\Omega)$ problem (1.7) *has a unique solution $\varphi \in H^{s,r}(\Omega) \cap L^\infty(\Omega)$, which admits the estimates*

$$\begin{gathered}\|\varphi\|_{H^{s,r}(\Omega)} \le C\sigma^{-1}\|f\|_{H^{s,r}(\Omega)} + C\sigma^{-1+\alpha}\|f\|_{L^\infty(\Omega)} \text{ for } sr \ne 1, 2,\\ \|\varphi\|_{H^{s,r}(\Omega)} \le C\sigma^{-1}\|f\|_{H^{s,r}(\Omega)} + C\sigma^{-1+\alpha}(1+\log\sigma)^{1/r}\|f\|_{L^\infty(\Omega)} \text{ for } sr = 1, 2,\end{gathered} \tag{1.13}$$

where the accretivity defect α is determined by

$$\alpha(r, s) = \max\big\{0,\ s - r^{-1},\ 2s - 3r^{-1}\big\}. \tag{1.14}$$

In order to obtain strong continuous solutions we introduce the scale of Banach spaces $X^{s,r}$ defined by the following.

Definition 1.6. *For any exponents s and r satisfying the inequalities*

$$0 < s < 1, \quad sr > 3, \quad \kappa = 2s - 3r^{-1} < 1,$$

denote by $X^{s,r}$ the Banach space $H^{s,r}(\Omega) \cap H^{1,\rho}(\Omega)$ endowed with the norm

$$\|u\|_{s,r} = \|u\|_{H^{s,r}(\Omega)} + \|u\|_{H^{1,\rho}(\Omega)}.$$

Here the exponent ρ is defined by the relations

$$\rho = (1-\kappa)^{-1} \text{ for } \kappa \le 1/2, \quad \rho = 3/(2-\kappa) \text{ for } 1/2 \le \kappa < 1, \tag{1.15}$$

so that the couples (s, r) and $(1, \rho)$ have the common accretivity defect

$$\alpha(r, s) = \alpha(\rho, 1) = \kappa.$$

We also denote by $X^{1+s,r}$ the Banach space of all functions $\varphi : \Omega \mapsto \mathbb{R}$ having the finite norm

$$\|\varphi\|_{1+s,r} = \|\varphi\|_{s,r} + \|\nabla\varphi\|_{s,r}$$

Note that the embeddings $X^{s,r} \hookrightarrow C(\Omega)$ and $X^{1+s,r} \hookrightarrow C^1(\Omega)$ are compact. Theorem 1.5 implies the following result which proof is given in Section 9.

Theorem 1.7. *Let $\Gamma = cl\, \Sigma_{\text{in}} \cap cl\, (\Sigma_{\text{out}} \cup \Sigma_0)$ and $\mathbf{U} \in C^\infty(\partial\Omega)$ comply with Condition 1.4, a vector field $\mathbf{u}$ satisfies boundary condition* (1.11), *and exponents s, r satisfy the inequalities*

$$0 < s < 1, \quad sr > 6, \quad \kappa = 2s - 3r^{-1} < 1. \tag{1.16}$$

Furthermore, assume that $\|\mathbf{u}\|_{1+s,r} \le R$. Then there are positive constants $\sigma^* > 1$ and C depending only on Ω, $\mathbf{U}$, s, r,, and R such that for any $\sigma > \sigma^*$ and $f \in X^{s,r}$, problem (1.7) *has a unique solution $\varphi \in X^{s,r}$, which admits the estimate*

$$\|\varphi\|_{s,r} \le C\sigma^{-1}\|f\|_{s,r} + C\sigma^{-1+2\kappa}\|f\|_{L^r(\Omega)}. \tag{1.17}$$

Since the space $X^{s,r}$ is a Banach algebra, Theorem 1.7 along with the contraction mapping principle yields the following result on solvability of the adjoint problem

$$\mathscr{L}^*\varphi := -\operatorname{div}(\varphi\mathbf{u}) + \sigma\varphi = f \quad \text{in } \Omega, \quad \varphi = 0 \quad \text{on } \Sigma_{\text{out}}. \tag{1.18}$$

Theorem 1.8. *Let the manifold $\Gamma^* = cl\, \Sigma_{\text{out}} \cap cl\, (\Sigma_{\text{in}} \cup \Sigma_0)$ and $\mathbf{U} \in C^\infty(\partial\Omega)$ comply with Condition 1.4, a vector field $\mathbf{u}$ and exponents s, r meet all requirements of. Theorem 1.7. Then there are positive constants $\sigma^* > 1, C$ depending only on Ω, $\mathbf{U}$, s, r,,and R, such that: for any $\sigma > \sigma^*$ and $f \in X^{s,r}$, problem* (1.18) *has a unique solution $\varphi \in X^{s,r}$ satisfying inequality* (1.17).

Lame equations and Bergman projection. The main idea of our approach is to express $\operatorname{div}\mathbf{u}$ in terms of ϱ and ϑ, next to substitute this expression in the mass balance equations, and by doing so to reduce the original problem to a boundary value problem for the transport equation. This requires careful analysis of solutions to the boundary value problem for the Lame equations

$$\Delta\mathbf{v} + \lambda\nabla \operatorname{div}\mathbf{v} = \mathbf{F} \text{ in } \Omega, \quad \mathbf{v} = 0 \text{ on } \partial\Omega. \tag{1.19}$$

The question is: whether is it possible to obtain direct expression for $\operatorname{div}\mathbf{v}$ without solving equations (1.19). It is easily seen that that $\operatorname{div}\mathbf{v}$ satisfies the operator equations

$$(\mathbf{I} + \lambda\mathscr{A})(\operatorname{div}\mathbf{v}) = F, \text{ where } \mathscr{A} = \operatorname{div}\Delta^{-1}\nabla, \quad F = \operatorname{div}\Delta^{-1}\mathbf{F}.$$

where Δ^{-1} be the inverse to the Laplace operator defined by the equalities

$$\Delta^{-1}F =: v, \quad \Delta v = F \text{ in } \Omega, \quad v = 0 \text{ on } \partial\Omega. \tag{1.20}$$

Hence the problem is to find the effective representation for the resolvent $(\mathbf{I} + \lambda\mathscr{A})^{-1}$. The unexpected fact is a connection between $\mathscr{A}$ and the Bergman projection. Recall, [2], that the harmonic Bergman space $b^r(\Omega)$ is defined by $b^r(\Omega) = \{u :\in L^r(\Omega) : u \text{ is harmonic in } \Omega\}$. The harmonic Bergman projection $\mathscr{Q}$ is defined to be the orthogonal projection from $L^2(\Omega)$ onto $b^2(\Omega)$, and the harmonic reproducing kernel $Q(x, y)$ is the integral kernel of the projection $\mathscr{Q}$. The metric properties of the harmonic Bergam projection for balls and half-spaces were studied in [2], [30], [31]. However, the boundedness of the harmonic Bergman projection in $L^r(\Omega)$ for regular bounded domains was proved only recently in paper [15]. The following theorem on the decomposition of operator A, which proof is given in Section 10, is the second main result of this paper.

Theorem 1.9. *Let Ω be a bounded domain with C^∞ boundary and $s \in [0, \infty)$, $r \in (1, \infty)$. Then:*
(i) The operator $\mathscr{A} : H^{s,r}(\Omega) \mapsto H^{s,r}(\Omega)$ is bounded and its norm depends only on Ω and s, r.
(ii) There exists a bounded operator $\mathscr{K} : H^{s,r}(\Omega) \mapsto H^{s+1,r}(\Omega)$, which norm depends only on Ω and s, r, such that $\mathscr{A} = \mathbf{I} - \mathscr{Q}/2 + \mathscr{K}$.
(iii) The Bergman projection $\mathscr{Q} : H^{s,r}(\Omega) \mapsto H^{s,r}(\Omega)$ is bounded.

In particular, this theorem yields the representation $Q(x, y) = 2\nabla_x\nabla_y G(x, y) + o(x, y)$ with a regular kernel $o(x, y)$ in terms of the harmonic Green function $G(x, y)$. Notice that the classic theory gives the formula $Q \sim \Delta_x\Delta_y G_2(x, y)$, where G_2 is the biharmonic Green function.

Existence theory. We are now in a position to formulate the main result of this paper on solvability of problem (1.1). Let us consider the following boundary value problem for the incompressible Navier-Stokes equations

$$\begin{gathered}\Delta\mathbf{u}_0 - \nabla p_0 = k \operatorname{div}(\mathbf{u}_0 \otimes \mathbf{u}_0), \quad \operatorname{div}\mathbf{u}_0 = 0 \text{ in } \Omega,\\ \mathbf{u}_0 = \mathbf{U} \text{ on } \partial\Omega, \quad \Pi p_0 = p_0 \ .\end{gathered} \tag{1.21}$$

In our notations Π is the projection,

$$\Pi u = u - \frac{1}{\operatorname{meas}\Omega}\int\limits_\Omega u\,dx. \tag{1.22}$$

It is well known, see [7], that for each $\mathbf{U} \in C^\infty(\Omega)$ satisfying the orthogonality conditions

$$\int_{\partial\Omega} \mathbf{U}\cdot\mathbf{n}\,ds = 0 \tag{1.23}$$

and all sufficiently small k, this problem has a unique C^∞ – solution. The triple $(\varrho_0, \mathbf{u}_0, \vartheta_0) =: (1, \mathbf{u}_0, 0)$ can be regarded as an approximate solution to problem (1.1) for small Mach numbers. We impose the following restrictions on the domain Ω and the boundary data:

H1. *The vector field $\mathbf{U} \in C^\infty$ and the manifold*

$$\Gamma\cap\Gamma^* := \big\{cl(\Sigma_{\text{in}}) \cap cl(\Sigma_{\text{out}} \cup \Sigma_0)\big\} \bigcup \big\{cl(\Sigma_{\text{out}}) \cap cl(\Sigma_{\text{in}} \cup \Sigma_0)\big\} \tag{1.24}$$

satisfy emergent field condition (1.4). *Moreover,* $\mathbf{U}$ *also satisfies condition* (1.23).

H2 *The mass forces are potential, i.e.,* $\mathbf{g} = -\nabla\Phi \in C^\infty(\Omega)$. *The boundary value of the density is in the form*

$$g = 1 - \omega^{-1}(p_0 + \Phi). \tag{1.25}$$

The unperturbed pressure p_0 *and potential* Φ *satisfy the orthogonality condition*

$$\int_{\partial\Omega} (p_0 + \Phi)(\mathbf{U}\cdot\mathbf{n})\, dS = 0. \tag{1.26}$$

Conditions (1.25) *and* (1.26) *prevent the formation of boundary layer and developing singularities near the inlet as* $\omega \to \infty$.

H3. *Furthermore, we shall assume that the viscosity ratio* λ *and exponents* s, r *satisfy the conditions*

$$0 < s < 1, \quad sr > 6, \quad \kappa = 2s - 3r^{-1} < 1/12 \tag{1.27}$$

$$C_0|||\mathscr{Q}|||_{s,r} + |||\mathscr{Q}|||_t < 2^{-1}(\lambda + 2) \text{ for all } t \in [6, r], \tag{1.28}$$

where C *is a constant in Theorem 1.5 corresponding to the exponents* s, r *and the constant* $R = 2\|\mathbf{u}_0\|_{1+s,r}$, *the notation* $|||\mathscr{Q}|||_{s,r}$, $|||\mathscr{Q}|||_r$ *stands for the norms of the Bergman projection*

$$|||\mathscr{Q}|||_{s,r} =: \|\mathscr{Q}\|_{L(X^{s,r}, X^{s,r})}, \quad |||\mathscr{Q}|||_t =: \|\mathscr{Q}\|_{L(L^t(\Omega), L^t(\Omega))}.$$

The following theorem is the main result of this paper.

Theorem 1.10. *Assume that* Ω, $\mathbf{U}$, s, r, *and* $\lambda > 1$ *comply with conditions* (**H**1)-(**H**3). *Then there exist positive constants* k^*, ω^* *such that for each fixed* $k \in [0, k^*]$, *and all* $\omega > \omega^*$, *problem* (1.1) *has a solution* $(\mathbf{u}_\omega, \varrho_\omega, \vartheta_\omega) \in (X^{1+s,r})^3 \times X^{s,r} \times X^{1+s,r}$ *such that*

$$\|\mathbf{u}_\omega - \mathbf{u}_0\|_{1+s,r} + \|\varrho_\omega - 1\|_{s,r} + \|\vartheta_\omega\|_{1+s,r} \to 0 \text{ as } \omega \to \infty. \tag{1.29}$$

1.4. **Structure of the paper.** Now we can explain the organization of the paper. In Section 2 we derive the perturbation equations (2.2) for the deviations $(\mathbf{v}, \varphi, \vartheta)$ of the states variables from the limiting quantities $(\mathbf{u}_0, 1, 0)$. The aim is to solve problem (2.2) by an application of the Schauder fixed point theory. In this framework our considerations are focused on the study of linearized boundary value problem (2.3). In Section 3 we derive L^2 estimates for solutions to this problem . In the next section we employ Theorem 1.9 to reduce the linearized problem (2.3) to the following boundary value problem for the transport operator equation

$$\mathbf{u}\nabla\varphi + \sigma\Pi\varphi + \frac{\sigma}{\lambda+2}\mathscr{Q}\Pi\varphi = \frac{\sigma}{\lambda+2}\mathscr{R}\varphi + f \text{ in } \Omega, \quad \varphi = 0 \text{ on } \Sigma_{\text{in}}. \tag{1.30}$$

Here $\sigma = \omega/(\lambda + 1)$, the projection Π is determined by (1.22), $\mathscr{Q}$ is the harmonic Bergman projection, and $\mathscr{R}$ is unessential compact operator. In sections 5 and 6 we derive a priori estimates and prove the solvability of problem (1.30) .There are two differences between problems (1.30) and (1.7). The first is the presence of the Bergman projection in equation (1.30). We cope with this difficulty assuming that λ is sufficiently large. The second is the presence of the projection Π in (1.30). It is important to note that the behavior of solutions to problem (1.30) drastically differs from the behavior of solutions to problem (1.7). While for solutions to problem (1.7) the Fichera-Oleinik estimates gives $\|\varphi\|_{L^2(\Omega)} \sim \sigma^{-1}\|f\|_{L^2(\Omega)}$, for solutions to problem (1.30) we also have $\|\Pi\varphi\|_{L^2(\Omega)} \sim \sigma^{-1}\|f\|_{L^2(\Omega)}$ but $|(\mathbf{I} - \Pi)\varphi| \sim \|f\|_{L^2(\Omega)}$. The disparity between $\Pi\varphi$ and $(\mathbf{I} - \Pi)\varphi$ leads to the singularity $\|\mathbf{v}\|_{1+s,r} \sim \omega^\kappa$

as $\omega \to \infty$. This indicates the formation of a weak boundary layer near inlet for small Mach numbers. In section 7 we show that a singular component of solutions vanishes for the well- prepared data satisfying Condition **H 2**. In the next section we complete the proof of Theorem 1.10. The last two sections are devoted to the proof of Theorems 1.5 and 1.9.

At the end of the section we discuss shortly possible generalizations of the obtained results.

In the case $sr < 1$, when the trace of a function $\varphi \in H^{s,r}(\Omega)$ on the boundary of Ω is not defined, the accretivity defect is equal to 0. It seems that in this case the emergent fields conditions is not needed for the solvability of solution to problem 1.7 in $H^{s,r}(\Omega)$. If this conjecture is correct, then the statement Theorem 1.10 holds true when the only manifold Γ satisfies the emergent vector field condition.

The restriction on the viscosity ratio λ is essential for our approach. The possible way to cope with this difficulty is to apply the projections $\mathscr{Q}$ and $\mathbf{I}-\mathscr{Q}$ to both the sides of operator transport equation (1.30). As a result we obtain the system of two transport equations for the functions $\mathscr{Q}\varphi$ and $(\mathbf{I}-\mathscr{Q})\varphi$, which involves the commutatior $[\mathbf{u}\nabla, \mathscr{Q}]$ and does not contain the large parameter σ at nonlocal terms. The fisrt difficulty is that the boundary conditions for the functions $\mathscr{Q}\varphi$ and $(\mathbf{I}-\mathscr{Q})\varphi$ are unknown. The second is that the commutator $[\mathbf{u}\nabla, \mathscr{Q}]$ is bounded in Sobolev spaces if and only if $\mathbf{u}\cdot\mathbf{n} = 0$ at $\partial\Omega$. Hence this trick works properly in the case when $\partial\Omega = \Sigma_0$, but in general case the problem can not be resolved in the frame of classic theory of transport equations.

Theorem 1.10 deals with "well-prepared" boundary data, satisfying conditions **H**2 , but even in this case the solutions are not uniformly smooth for small Mach numbers. The investigation of this problen requires the construction of the formal asymptotics of solutions for large ω.

2. Perturbation equations. Linearized problem.

In this section we deduce the equations for the deviations of state variables $(\mathbf{u}, \varrho, \vartheta)$ from their limiting values. We shall look for a solution to problem (1.1) in the form

$$\mathbf{u} = \mathbf{u}_0 + \mathbf{v}, \quad \varrho = 1 + \omega^{-1}(p_0 + \Phi) + \varphi - \vartheta, \tag{2.1}$$

where $\mathbf{v}$ and φ are new unknown functions. Substituting this expressions into (1.1) leads to the following equations for the functions $(\mathbf{v}, \varphi, \vartheta)$,

$$\Delta\mathbf{v} + \lambda\nabla\,\mathrm{div}\,\mathbf{v} - \omega\nabla\varphi - k\mathscr{U}\mathbf{v} = \Psi[\mathbf{v}, \varphi, \vartheta] \tag{2.2a}$$

$$\mathbf{u}\nabla\varphi + \mathrm{div}\,\mathbf{v} - \mathrm{div}(\mathbf{u}\vartheta) = \Upsilon[\mathbf{v}, \varphi, \vartheta], \tag{2.2b}$$

$$\Delta\vartheta - k\mathbf{u}\nabla\vartheta + kb\,\mathrm{div}(\mathbf{u}\Pi\varphi) - k\omega^{-1}\mathscr{W}\mathbf{v} = \Theta[\mathbf{v}, \varphi, \vartheta], \tag{2.2c}$$

$$\mathbf{v} = 0, \quad \vartheta = 0 \text{ on } \partial\Omega, \quad \varphi = 0 \text{ on } \Sigma_{\mathrm{in}}. \tag{2.2d}$$

Here $b = 1 - \gamma^{-1}$, the differential operators $\mathscr{U}$ and $\mathscr{W}$ are defined by the equalities

$$\begin{aligned} &\mathscr{U}\mathbf{v} = \mathrm{div}(\mathbf{a}\otimes\mathbf{v} + \mathbf{v}\otimes\mathbf{a}), \quad \mathbf{a} = \mathbf{u}_0 + 2^{-1}\mathbf{v}, \\ &\mathscr{W}\mathbf{v} = -2b\,\mathrm{div}\left((\nabla\mathbf{u}_0 + \nabla\mathbf{u}_0^*)\mathbf{v}\right) + 2b\Delta\mathbf{u}_0\cdot\mathbf{v}, \end{aligned} \tag{2.2e}$$

operators Ψ, Υ, and Θ are given by

$$\begin{aligned}\Psi = k\operatorname{div}(\varsigma\mathbf{u}\otimes\mathbf{u}) + \omega\nabla(\varsigma\vartheta) - \varsigma\nabla\Phi, \quad \Upsilon = \Upsilon_1+\Upsilon_2, \quad \Theta = \Theta_1+\Theta_2,\\ \Upsilon_1 = -\omega^{-1}\operatorname{div}((p_0+\Phi)\mathbf{u}), \quad \Upsilon_2 = -\varphi\operatorname{div}\mathbf{v},\\ \Theta_1 = -\omega^{-1}kb\Big(\mathbf{u}\nabla(p_0+\Phi)+D_0\Big),\\ \Theta_2 = kb\Big((\Pi\varphi+\vartheta+\varsigma\vartheta)\operatorname{div}\mathbf{v}+\omega^{-1}D_1\Big)+k(1-b)\varsigma\mathbf{u}\nabla\vartheta,\end{aligned} \tag{2.2f}$$

where

$$\varsigma = \varphi-\vartheta+\omega^{-1}(p_0+\Phi), \quad 2D_0 = (\nabla\mathbf{u}_0+\nabla\mathbf{u}_0^*)^2, \quad 2D_1 = (\nabla\mathbf{v}+\nabla\mathbf{v}^*)^2+2(\lambda-1)\operatorname{div}\mathbf{v}^2.$$

Our aim is to prove the existence of small solution to this problem for all sufficiently large ω. In this framework, considerations are focused on the analysis of the linearized problem

$$\Delta\mathbf{v}+\lambda\nabla\operatorname{div}\mathbf{v}-\omega\nabla\varphi = k\mathscr{U}\mathbf{v}+\mathbf{F} \text{ in } \Omega, \tag{2.3a}$$

$$\mathbf{u}\nabla\varphi+\operatorname{div}\mathbf{v}-\operatorname{div}(\vartheta\mathbf{u}) = G \text{ in } \Omega, \tag{2.3b}$$

$$\Delta\vartheta - k\mathbf{u}\nabla\vartheta + kb\operatorname{div}(\mathbf{u}\Pi\varphi) = k\omega^{-1}\mathscr{W}\mathbf{v}+H, \text{ in } \Omega \tag{2.3c}$$

$$\varphi = 0 \text{ on } \Sigma_{\text{in}}, \quad \mathbf{v}=0, \quad \vartheta = 0 \text{ on } \partial\Omega, \tag{2.3d}$$

where $\mathbf{u}$ and $\mathbf{a}$ are considered as given functions of class $X^{1+s,r}$. With applications to nonlinear problem (2.2) in mind, we shall assume that functions $\mathbf{F}$ and H admit the representations

$$\mathbf{F} = \operatorname{div}\mathbb{F}+\mathbf{f}, \quad H = \operatorname{div}\mathbf{h}+h_0 \tag{2.4}$$

in which a matrix-valued function $\mathbb{F}:\Omega\mapsto\mathbb{R}^9$, vector fields $\mathbf{f},\mathbf{h}:\Omega\mapsto\mathbb{R}^3$, and a function $h_0:\Omega\mapsto\mathbb{R}$ have the finite the norm

$$|\mathbf{F}|_t = \|\mathbf{f}\|_{L^t(\Omega)}+\|\mathbb{F}\|_{L^t(\Omega)}, \quad |H|_t = \|\mathbf{h}\|_{L^t(\Omega)}+\|h_0\|_{L^t(\Omega)}, \tag{2.5}$$

$$|\mathbf{F}|_{s,r} = \|\mathbf{f}\|_{s,r}+\|f_0\|_{s,r}, \quad |H|_{s,r} = \|\mathbf{h}\|_{s,r}+\|h_0\|_{s,r}, \tag{2.6}$$

Hence our first goal is to prove the well posedness of problem (2.3). The proof occupies the next four sections.

3. Linear problem. First estimates

In this section we derive L^2 estimates for solutions to linear problem (2.3). By abuse of notation, we will write $\overline{\varphi}$ and m instead of $\Pi\varphi$ and $(\mathbf{I}-\Pi)\varphi$.

Theorem 3.1. *Let a vector field $\mathbf{u}\in X^{1+s,r}$ and exponents s,r meet all requirements of Theorem 1.10, and $\|\mathbf{u}\|_{1+s,r}+\|\mathbf{a}\|_{1+s,r}\le R$. Then for any $\varepsilon>0$, there exist constants c, $k^*>0$, and $\omega^*>1$, depending only on $\partial\Omega$, s, r, R, and ε, such that for all $\omega>\omega^*$ and $k\in[0,k^*]$ a solution $(\mathbf{v},\varphi,\vartheta)\in(X^{1+s,r})^3\times X^{s,r}\times X^{1+s,r}$ to problem* (2.3) *satisfies the inequalities*

$$\|\mathbf{v}\|_{H^{1,2}(\Omega)} \le c|\mathbf{F}|_2 + c\omega^{1/2+\varepsilon}(\|G\|_{L^2(\Omega)}+|H|_2), \tag{3.1a}$$

$$\|\overline{\varphi}\|_{L^2(\Omega)} \le c\omega^{-1}|\mathbf{F}|_2 + c\omega^{-1/2+\varepsilon}(\|G\|_{L^2(\Omega)}+|H|_2), \tag{3.1b}$$

$$|m| \le c\omega^{-1/2+\varepsilon}|\mathbf{F}|_2 + c\omega^{\varepsilon}(\|G\|_{L^2(\Omega)}+|H|_2), \tag{3.1c}$$

$$\|\vartheta\|_{H^{1,2}(\Omega)} \le c(\omega^{-1}k|\mathbf{F}|_2 + k\omega^{-1/2+\varepsilon}\|G\|_{L^2(\Omega)}+|H|_2), \tag{3.1d}$$

where the norm $|\cdot|_2$ is defined by (2.5).

We split the proof into three steps. First we employ the energy identity to obtain the estimates for the velocity vector field and $\overline{\varphi}$ via the average density m, next we apply Theorem 1.7 to obtain the estimate for m, and finally we deduce inequalities (3.1).

Step 1. Denote by $\mathbf{P}$ and S the functions

$$\begin{aligned} \mathbf{P} \equiv k\mathscr{U}\mathbf{v} + \mathbf{F} =: \operatorname{div}\mathbb{P} + \mathbf{p}, \quad S \equiv \frac{k}{\omega}\mathscr{W}\mathbf{v} + H =: \operatorname{div}\mathbf{s} + s_0, \\ \mathbb{P} = k\mathbf{a}\otimes\mathbf{v} + k\mathbf{v}\otimes\mathbf{a} + \mathbb{F}, \quad \mathbf{p} = \mathbf{f}, \\ \mathbf{s} = -\frac{2kb}{\omega}(\nabla\mathbf{u}_0 + \nabla\mathbf{u}_0^*)\mathbf{v} + \mathbf{h}, \quad s_0 = \frac{2kb}{\omega}\Delta\mathbf{u}_0\cdot\mathbf{v} + h_0. \end{aligned} \tag{3.2}$$

Multiplying both sides of equation (2.3a) by $\mathbf{v}$, integrating the result by parts, and recalling equation (2.3b) we arrive at the identity

$$\int_\Omega \big(|\nabla\mathbf{v}|^2 + \lambda|\operatorname{div}\mathbf{v}|^2\big)\,dx = -\langle\mathbf{v},\mathbf{P}\rangle + \omega\langle\varphi,\operatorname{div}\mathbf{v}\rangle,$$

where $\langle\cdot,\cdot\rangle$ denotes the scalar product in $L^2(\Omega)$. On the other hand, equation (2.3b) implies

$$\int_\Omega \varphi\operatorname{div}\mathbf{v}\,dx = \langle\varphi, G + \operatorname{div}(\vartheta\mathbf{u})\rangle + \frac{1}{2}\int_\Omega \varphi^2\operatorname{div}\mathbf{u}\,dx - \frac{1}{2}\int_{\Sigma_{\text{out}}} \varphi^2|\mathbf{U}_n|\,d\Sigma.$$

By virtue of orthogonality condition (1.23), we have

$$\frac{1}{2}\int_\Omega \varphi^2\operatorname{div}\mathbf{u}\,dx = m\langle\overline{\varphi},\operatorname{div}\mathbf{u}\rangle + \frac{1}{2}\int_\Omega \overline{\varphi}^2\operatorname{div}\mathbf{u}\,dx.$$

Combining these identities we obtain

$$\begin{aligned} \int_\Omega \big(|\nabla\mathbf{v}|^2 + \lambda|\operatorname{div}\mathbf{v}|^2\big)\,dx + \frac{\omega}{2}\int_{\Sigma_{\text{out}}} |\mathbf{U}_n|\varphi^2\,d\Sigma = -\langle\mathbf{v},\mathbf{P}\rangle + \\ \omega m\big[\langle 1, G + \operatorname{div}(\vartheta\mathbf{u})\rangle + \langle\overline{\varphi},\operatorname{div}\mathbf{u}\rangle\big] + \omega\langle\overline{\varphi}, G + \operatorname{div}(\vartheta\mathbf{u})\rangle + \frac{\omega}{2}\langle\operatorname{div}\mathbf{u},\overline{\varphi}^2\rangle. \end{aligned} \tag{3.3}$$

Since $\mathbf{v}$ vanishes on $\partial\Omega$, we have for any $\delta > 0$, $|\langle\mathbf{v},\mathbf{P}\rangle| \le \delta\|\nabla\mathbf{v}\|^2_{L^2(\Omega)} + c\delta^{-1}|\mathbf{P}|_2^2$, where c depends only on Ω. Recalling the estimate $\|\mathbf{u}\|_{C^1(\Omega)} \le cR$ we conclude from this and (3.3) that for suitable choice of k,

$$\begin{aligned} \|\mathbf{v}\|^2_{H^{1,2}(\Omega)} \le c|\mathbf{P}|_2^2 + c\omega|m|(\|G\|_{L^2(\Omega)} + \|\vartheta\|_{H^{1,2}(\Omega)} + \|\overline{\varphi}\|_{L^2(\Omega)}) + \\ c\omega(\|G\|^2_{L^2(\Omega)} + \|\vartheta\|^2_{H^{1,2}(\Omega)} + \|\overline{\varphi}\|^2_{L^2(\Omega)}), \end{aligned} \tag{3.4}$$

where the constant c depends only on Ω, R , and k. Next lemma gives the estimate for ϑ in terms of the deviation $\overline{\varphi} = \varphi - m$.

Lemma 3.2. *There exists k^*, depending only on $\|\mathbf{u}\|_{C^1(\Omega)}$, such that for all $r \in (1,\infty)$ and $k \in [0,k^*]$, a solution to problem* (2.3c)- (2.3d) *admits the estimate*

$$\|\vartheta\|_{H^{1,2}(\Omega)} \le c(\Omega)(k\|\overline{\varphi}\|_{L^2(\Omega)} + |S|_2). \tag{3.5}$$

Proof. The proof obviously follows from Lemma 1.1. □

Combining (3.5) with inequality (3.4) we obtain the estimate

$$\|\mathbf{v}\|^2_{H^{1,2}(\Omega)} \leq c\omega|m|(E + \|\overline{\varphi}\|_{L^2(\Omega)}) + c\omega(\|\overline{\varphi}\|^2_{L^2(\Omega)} + E^2) + |\mathbf{P}|_2^2, \tag{3.6}$$

where $E = \|G\|_{L^2(\Omega)} + |S|_2$.
The next lemma gives the complementary estimate for $\overline{\varphi}$ in terms of $\mathbf{v}$ and ϑ.

Lemma 3.3. *Under the assumptions of Theorem 3.1 there is $k^* > 0$ depending only on R and Ω such that for $k \in [0, k^*]$, each solution to problem* (2.3) *admits the estimate*

$$\|\overline{\varphi}\|_{L^2(\Omega)} \leq c(\Omega)\omega^{-1}(\|\mathbf{v}\|_{H^{1,2}(\Omega)} + |\mathbf{P}|_2) \tag{3.7}$$

Proof. Choose an arbitrary function $\zeta \in L^2(\Omega)$ with $\Pi\zeta = \zeta$ and a vector field $\mathbf{q}$ such that

$$\mathbf{q} \in H^{1,2}_0(\Omega), \quad \operatorname{div} \mathbf{q} = \zeta, \quad \|\mathbf{q}\|_{H^{1,2}(\Omega)} \leq c(\Omega)\|\zeta\|_{L^2(\Omega)}.$$

Multiplying both sides of equations (2.3a) by $\mathbf{q}$ and integrating the result over Ω we obtain

$$\omega \int\limits_\Omega \varphi\zeta\, dx = \int\limits_\Omega (\nabla \mathbf{v} : \nabla \mathbf{q} + \lambda \operatorname{div} \mathbf{v} \operatorname{div} \mathbf{q})\, dx + \langle \mathbf{P}, \mathbf{q}\rangle,$$

which yields (3.7) and the lemma follows. □

Step 2. In this paragraph we estimate the quantity m related to the mean value of the "density" φ. Our considerations are based on the following auxiliary lemmas, first of which constitutes the continuity of the embedding $X^{s,r} \hookrightarrow H^{1/2,2}(\Omega)$.

Lemma 3.4. *There is a constant c, depending only on Ω and s, r, such that the inequality $\|f\|_{H^{1/2,2}(\Omega)} \leq c\|f\|_{X^{s,r}}$ holds true for all functions $f \in X^{s,r}$.*

Proof. Since by virtue of (1.3) the embedding $H^{1,\rho}(\Omega) \hookrightarrow H^{1/2,2}(\Omega)$ is bounded for all $\rho \geq 3/2$, it suffices to prove the lemma for the case when the exponent $\rho(r,s)$ in Definition 1.6 of the space $X^{s,r}$ satisfies the inequality $\rho < 3/2$ and, consequently, $\kappa = 2s - 3r^{-1}$ satisfies the inequality $\kappa < 1/3$. By virtue of (1.15) in this case we have $\rho = (1-\kappa)^{-1}$. Introduce the quantities

$$\tau = \frac{2-\rho}{r-\rho}, \quad \nu = rs\tau \equiv \frac{(\kappa r + 3)(1-2\kappa)}{2((1-\kappa)r - 1)}$$

By Definition 1.6, we have $r > 3$ and hence $\tau \in (0,1)$. The inequality $(r/3)(2(1-\kappa) - \kappa(1-2\kappa)) > 5/3 - 2\kappa$, which is obviously true for all$r > 3$ and $\kappa \in [0, 1/3]$, implies the inclusion $\nu \in (0,1)$.

Now choose an arbitrary function f with $\|f\|_{X^{s,r}} = 1$, and note that

$$\begin{aligned}|x-y|^{-1}|f(x)-f(y)|^2 &= \big((|x-y|^{-\frac{\nu}{\tau}}|f(x)-f(y)|^r\big)^\tau \big((|x-y|^{-\frac{1-\nu}{1-\tau}}|f(x)-f(y)|^\rho\big)^{1-\tau} \\ &\leq c|x-y|^{-\frac{\nu}{\tau}}|f(x)-f(y)|^r + c|x-y|^{-\frac{1-\nu}{1-\tau}}|f(x)-f(y)|^\rho = \\ &\quad |x-y|^{-rs}|f(x)-f(y)|^r + c|x-y|^{-q\rho}|f(x)-f(y)|^\rho,\end{aligned}$$

where

$$q = \frac{1-\nu}{\rho(1-\tau)} = \frac{1-rs\tau}{\rho(1-\tau)} < \frac{1-3\tau}{\rho(1-\tau)} < 1.$$

Thus, we get the following estimate for the semi-norm $|f|_{1/2,2,\Omega}$ defined by (1.2)

$$|f|^2_{1/2,2\Omega} \leq c|f|^r_{s,r\Omega} + c|f|^\rho_{q,\rho,\Omega}.$$

¿From this and Definition 1.6 of the space $X^{s,r}$ we conclude that

$$\|f\|^2_{H^{1/2,2}(\Omega)} \le c\|f\|^r_{H^{s,r}(\Omega)} + \|f\|^\rho_{H^{q,\rho}(\Omega)} \le c\|f\|^r_{H^{s,r}(\Omega)} + \|f\|^\rho_{H^{1,\rho}(\Omega)} \le c(\|f\|^r_{s,r} + \|f\|^\rho_{s,r}) \le c,$$

and the proof lemma 3.4 is completed. □

Lemma 3.5. *Let exponents s, r and a vector field $\mathbf{u} \in X^{1+s,r}$ meet all requirements of Theorem 1.8. Furthermore, assume that $\|\mathbf{u}\|_{1+s,r} \le R$. Then there are $\sigma_1 > 1$, $c > 0$ and $\delta > 0$, depending only on Ω, $\mathbf{U}$,s, r, and R such that the adjoint boundary value problem*

$$-\operatorname{div}(\eta\mathbf{u}) + \sigma_1\eta = \sigma_1 \text{ in } \Omega, \quad \eta = 0 \text{ on } \Sigma_{\text{out}} \tag{3.8}$$

has a solution, satisfying the inequalities

$$\|\eta\|_{s,r} \le c, \quad \sigma_1 \int_\Omega (1-\eta)\,dx > \delta > 0. \tag{3.9}$$

Proof. Choose σ^* so large that conditions of Theorem 1.8 are fulfilled for the couple of exponents (s, r). Fix an arbitrary $\sigma_1 > \max\{\sigma^*, \|\operatorname{div}\mathbf{u}\|_{C^1(\Omega)}\}$. Recall that $\|\mathbf{u}\|_{C^1(\Omega)} \le c(s, r)R$. By virtue of Theorem 1.8, problem (3.8) has a unique solution $\eta \in X^{s,r}$ satisfying the inequality $\|\eta\|_{s,r} \le c(r, s, \sigma_1)$. Let us show that η is non-negative. Multiplying both sides of equation (3.8) by the function $\eta_- = \min\{0, \eta\}$ and integrating the result over Ω we obtain the identity

$$\int_\Omega (\sigma_1 - 2^{-1}\operatorname{div}\mathbf{u})\eta_-^2\,dx - 2^{-1}\int_{\Sigma_{\text{in}}} (\mathbf{U}\mathbf{n})\eta_-^2\,d\Sigma = \sigma_1\int_\Omega \eta_-\,dx \le 0,$$

which yields $\eta_- = 0$ and $\eta \ge 0$.

Next we show that η is strictly positive on Σ_{in}. Since η is a continuous strong solution to equation (3.8) and the derivatives of the vector field $\mathbf{u}$ are continuous in cl Ω, the function $\mathbf{u}\cdot\nabla\eta$ is continuous in cl Ω. Hence η is continuously differentiable along integral lines of vector field $\mathbf{u}$. Choose an arbitrary point $P \in \Sigma_{\text{in}}$. If $\eta(P) = 0$, then $\mathbf{u}\cdot\nabla\eta(P) = -\sigma_1 < 0$. It follows from this and the inequality $\mathbf{u}(P)\cdot\mathbf{n}(P) < 0$ that $\eta < 0$ on some segment of the integral line, passing through P and belonging to Ω, which contradicts non-negativity of η. Hence $\eta(P) > \delta(P) > 0$ at each point $P \in \Sigma_{\text{in}}$. Let us prove that δ does not depend on the choice of a solution and a vector field $\mathbf{u}$. Assume, in contrary to our claim, that there exists a sequence of vector fields $\mathbf{u}_n$ satisfying the conditions of Lemma 3.5 and a sequence η_n of solutions to problem (3.8) (with $\mathbf{u}$ replaced by $\mathbf{u}_n$) such that $\eta_n(P) \to 0$ as $n \to \infty$. Without any loss of generality we can assume that $\mathbf{u}_n \to \mathbf{u}$ weakly in $X^{1+s,r}$ and $\eta_n \to \eta$ weakly in $X^{s,r}$. In particular, $\mathbf{u}_n \to \mathbf{u}$ in $C^1(\Omega)$, and $\eta_n \to \eta$ in $C(\Omega)$. It follows from this that η is a strong continuous solution to problem (3.8) and, as it was mentioned above, $\eta(P) > 0$, which contradict to equality $\eta(P) = \lim \eta_n(P) = 0$. Hence for each $P \in \Sigma_{\text{in}}$, we have $\eta(P) > \delta(P) > 0$, where δ depends only on P, s, r, Ω, R and $\mathbf{U}$. Combining this result with the identity

$$\sigma_1\int_\Omega (1-\eta)\,dx = -\int_{\Sigma_{\text{in}}} (\mathbf{U}\cdot\mathbf{n})\eta(P)\,d\Sigma$$

we obtain (3.9), which completes the proof. □

We are now in a position to estimate the quantity m. Multiplying both sides of (2.3b) by η and integrating the result by parts we get the integral identity

$$m\sigma_1 \int_\Omega (1-\eta)\,dx = \int_\Omega \eta\mathfrak{G}\,dx - \sigma_1 \int_\Omega (1-\eta)\overline{\varphi}\,dx,$$

where

$$\mathfrak{G} = G + \operatorname{div}(\vartheta\mathbf{u}) - \operatorname{div}\mathbf{v}$$

Since, by virtue of Lemma 3.4, $\|\eta\|_{H^{1/2,2}(\Omega)} \le c$, we have

$$|m| \le c\|\overline{\varphi}\|_{L^2(\Omega)} + c \sup_{\|\zeta\|_{H^{1/2,2}(\Omega)}=1} \int_\Omega \zeta\mathfrak{G}\,dx. \tag{3.10}$$

On the other hand, multiplying both sides of (2.3b) by $\zeta \in H^{1,2}_0(\Omega)$ and integrating the result by parts we get the identity

$$\int_\Omega \zeta\mathfrak{G}\,dx = \int_\Omega \operatorname{div}(\zeta\mathbf{u})\overline{\varphi}\,dx,$$

which leads to the estimate

$$\|\mathfrak{G}\|_{H^{-1,2}(\Omega)} \le c(\Omega, R)\|\overline{\varphi}\|_{L^2(\Omega)}. \tag{3.11}$$

Next note that for all $0 \le \beta < 1/2$, we have $H^{-\beta,2}(\Omega) = H^{\beta,2}(\Omega)'$, which together with the interpolation theorem and the embedding $H^{1,2}(\Omega) \hookrightarrow H^{\beta,2}(\Omega)$ implies the inequalities

$$\sup_{\|\zeta\|_{H^{1/2,2}(\Omega)}=1} \int_\Omega \zeta\mathfrak{G}\,dx \le c \sup_{\|\zeta\|_{H^{\beta,2}(\Omega)}=1} \int_\Omega \zeta\mathfrak{G}\,dx = \|\mathfrak{G}\|_{H^{\beta,2}(\Omega)'} =$$
$$\|\mathfrak{G}\|_{H^{-\beta,2}(\Omega)} \le c(\beta,\Omega)(\|\mathfrak{G}\|_{H^{-1,2}(\Omega)})^\beta(\|\mathfrak{G}\|_{L^2(\Omega)})^{1-\beta}.$$

Combining this estimate with (3.11) we obtain

$$|m| \le c(\|\overline{\varphi}\|_{L^2(\Omega)} + c(\|\overline{\varphi}\|_{L^2(\Omega)})^\beta\,\|\mathfrak{G}\|^{1-\beta}_{L^2(\Omega)}$$

Next note that by virtue of inequality (3.5) we have

$$\|\mathfrak{G}\|_{L^2(\Omega)} \le c(\|\overline{\varphi}\|_{L^2(\Omega)} + E + \|\mathbf{v}\|_{H^{1,2}(\Omega)}),$$

which implies the estimate

$$|m| \le c\|\overline{\varphi}\|_{L^2(\Omega)} + c\|\overline{\varphi}\|^\beta_{L^2(\Omega)}\,(\|\overline{\varphi}\|_{L^2(\Omega)} + E + \|\mathbf{v}\|_{H^{1,2}(\Omega)})^{1-\beta} \tag{3.12}$$

Step 3. We begin with the observation that inequality (3.12) leads to the estimate

$$\omega|m|(\|\overline{\varphi}\|_{L^2(\Omega)} + E) \le c\omega\|\overline{\varphi}\|_{L^2(\Omega)}\big(\|\overline{\varphi}\|_{L^2(\Omega)} + E\big) +$$
$$c\omega\|\overline{\varphi}\|^\beta_{L^2(\Omega)}\big(\|\overline{\varphi}\|_{L^2(\Omega)} + E\big)^{2-\beta} + c\omega\|\overline{\varphi}\|^\beta_{L^2(\Omega)}\|\mathbf{v}\|^{1-\beta}_{H^{1,2}(\Omega)}\big(\|\overline{\varphi}\|_{L^2(\Omega)} + E\big)$$

¿From this we conclude that

$$\begin{aligned}\omega\,|m|\,(\|\overline{\varphi}\|_{L^2(\Omega)} + E) \le c\omega\,\big(\|\overline{\varphi}\|^2_{L^2(\Omega)} + E^2\big) + \\ c\omega\|\overline{\varphi}\|^{1+\beta}_{L^2(\Omega)}\,\|\mathbf{v}\|^{1-\beta}_{H^{1,2}(\Omega)} + c\omega\,E\,\|\overline{\varphi}\|^\beta_{L^2(\Omega)}\,\|\mathbf{v}\|^{1-\beta}_{H^{1,2}(\Omega)}.\end{aligned} \tag{3.13}$$

By virtue of Lemma 3.3 we have

$$\|\overline{\varphi}\|_{L^2(\Omega)} \le c\omega^{-1}(\|\mathbf{v}\|_{H^{1,2}(\Omega)} + |\mathbf{P}|_2), \tag{3.14}$$

which along with the Young inequality leads to the estimates

$$\omega\|\overline{\varphi}\|^{1+\beta}_{L^2(\Omega)}\,\|\mathbf{v}\|^{1-\beta}_{H^{1,2}(\Omega)} \le c\omega^{-\beta}(\|\mathbf{v}\|^2_{H^{1,2}(\Omega)} + \|\mathbf{v}\|^{1+\beta}_{H^{1,2}(\Omega)}|\mathbf{P}|_2^{1-\beta}) \le c\omega^{-\beta}(\|\mathbf{v}\|^2_{H^{1,2}(\Omega)} + |\mathbf{P}|_2^2). \tag{3.15}$$

Repeating these arguments gives the inequality

$$\omega\, E\, \|\overline{\varphi}\|^{\beta}_{L^2(\Omega)}\,\|\mathbf{v}\|^{1-\beta}_{H^{1,2}(\Omega)} \le c\omega^{1-\beta}E(\|\mathbf{v}\|_{H^{1,2}(\Omega)} + |\mathbf{P}|_2).$$

By the Cauchy inequality, for any positive δ,

$$\omega^{1-\beta}E\,(\|\mathbf{v}\|_{H^{1,2}(\Omega)} + |\mathbf{P}|_2) \le \delta(\|\mathbf{v}\|^2_{H^{1,2}(\Omega)} + |\mathbf{P}|_2^2) + c(\delta)\omega^{2-2\beta}E^2.$$

¿From this we conclude that

$$\omega\, E\, \|\overline{\varphi}\|^{\beta}_{L^2(\Omega)}\,\|\mathbf{v}\|^{1-\beta}_{H^{1,2}(\Omega)} \le \delta\|\mathbf{v}\|^2_{H^{1,2}(\Omega)} + c(\delta)\omega^{2-2\beta}E^2 + c|\mathbf{P}|_2^2 \tag{3.16}$$

Next, inequality (3.14) yields the estimate

$$\omega\left(\|\overline{\varphi}\|^2_{L^2(\Omega)} + E^2\right) \le c\omega^{-1}(\|\mathbf{v}\|^2_{H^{1,2}(\Omega)} + |\mathbf{P}|_2^2) + c\omega E^2 \tag{3.17}$$

Substituting (3.15), (3.16), and (3.17) into (3.13) and noting that $\omega > 1$, $\omega \le \omega^{2-2\beta}$ we obtain the inequality

$$\omega\,|m|\,(\|\overline{\varphi}\|_{L^2(\Omega)} + E) \le (\delta + c\omega^{-\beta})\,\|\mathbf{v}\|^2_{H^{1,2}(\Omega)} + c(\delta)\omega^{2(1-\beta)}E^2 + c|\mathbf{P}|_2^2$$

In its turn, substituting this inequality into (3.6) and using inequality (3.14) we arrive at the estimate

$$(1 - c\delta - c\omega^{-\beta})\,\|\mathbf{v}\|^2_{H^{1,2}(\Omega)} \le c|\mathbf{P}|_2^2 + c\omega^{2-2\beta}E^2,$$

where a constant c depends only on Ω,β and $\|\mathbf{u}\|_{C^1(\Omega)}$. From this we conclude that for all $\beta \in (0, 1/2)$ and $\omega > \omega^*(\Omega, \beta, \|\mathbf{u}\|_{C^1(\Omega)})$, a solution to problem (2.3) satisfies the inequality

$$\|\mathbf{v}\|^2_{H^{1,2}(\Omega)} \le c|\mathbf{P}|_2^2 + c\omega^{2-2\beta}E^2, \tag{3.18}$$

which together with (3.14) implies the estimate

$$\|\overline{\varphi}\|_{L^2(\Omega)} \le c\omega^{-1}|\mathbf{P}|_2 + c\omega^{-\beta}E\ . \tag{3.19}$$

Combining inequalities (3.18),(3.19), and (3.12) we arrive at the estimate

$$|m| \le c(\omega^{-1}|\mathbf{P}|_2 + \omega^{-\beta}E)\ . \tag{3.20}$$

Finally inequalities (3.19) and (3.5) lead to the estimate for the temperature

$$\|\vartheta\|_{H^{1,2}(\Omega)} \le c\omega^{-1}k|\mathbf{P}|_2 + ck\omega^{-\beta}\|G\|_{L^2(\Omega)} + c|H|_2. \tag{3.21}$$

Notice that for $\beta = (1-\varepsilon)/2$, inequalities (3.18)-(3.21) imply the estimates

$$\begin{aligned}
\|\mathbf{v}\|_{H^{1,2}(\Omega)} &\le c|\mathbf{P}|_2 + c\omega^{1/2+\varepsilon}(\|G\|_{L^2(\Omega)} + |S|_2),\\
\|\overline{\varphi}\|_{L^2(\Omega)} &\le c\omega^{-1}|\mathbf{P}|_2 + c\omega^{-1/2+\varepsilon}(\|G\|_{L^2(\Omega)} + |S|_2),\\
|m| &\le c\omega^{-1/2+\varepsilon}|\mathbf{P}|_2 + c\omega^{\varepsilon}(\|G\|_{L^2(\Omega)} + |S|_2),\\
\|\vartheta\|_{H^{1,2}(\Omega)} &\le c(\omega^{-1}k|\mathbf{P}|_2 + k\omega^{-1/2+\varepsilon}\|G\|_{L^2(\Omega)} + |S|_2)\ .
\end{aligned} \tag{3.22}$$

On the other hand, representation (3.2) yields

$$|\mathbf{P}|_2 \le |\mathbf{F}|_2 + ck\|\mathbf{v}\|_{H^{1,2}(\Omega)}, \quad |S|_2 \le |H|_2 + ck\omega^{-1}\|\mathbf{v}\|_{H^{1,2}(\Omega)}.$$

Substituting this result into (3.22) and choosing k and ω^{-1} sufficiently small we obtain (3.1), which completes the proof of Theorem 3.1.

4. Lame equation and Bergman projection

Now our aim is to derive a priori estimates in Sobolev spaces for solutions to linear problem (2.3). To this end we eliminate $\operatorname{div}\mathbf{v}$ from the mass balance equations (2.3b), and convert equations (2.3) into a system of transport equation for φ and elliptic equations for $\mathbf{v}$ and ϑ. This procedure requires the detailed analysis of the boundary value problem

$$\Delta\mathbf{v} + \lambda\nabla\operatorname{div}\mathbf{v} = \omega\nabla\varphi + \mathbf{P} \text{ in } \Omega, \quad \mathbf{v} = 0 \text{ on } \partial\Omega. \tag{4.1}$$

In this section we establish the solvability of problem (4.1) and discuss in details the relation between the Lame operator and the harmonic Bergman projection. Further we shall assume that $\Omega \subset \mathbb{R}^3$ is a bounded domain with C^∞ boundary $\partial\Omega$. Let Δ^{-1} be the inverse to the Laplace operator defined by (1.20). The following result is a straightforward consequence of Lemma 1.1.

Lemma 4.1. *For any $r \in (1,\infty)$ and $s \in [0,\infty)$, the operators $\Delta^{-1} : H^{s,r}(\Omega) \to H^{s+2,r}(\Omega)$, $\Delta^{-1}\operatorname{div} : H^{s,r}(\Omega) \to H^{s+1,r}(\Omega)$ are bounded.*

The main result of this section is the following theorem. Introduce the operator $\mathscr{T}$ and parameter σ defined by the equalities

$$\mathscr{T} = (\lambda+1)^{-1}\big(\mathbf{I} + \lambda(\lambda+2)^{-1}\mathscr{Q}\big), \quad \sigma = \omega(1+\lambda)^{-1}.$$

By Theorem 1.9, the operators $\mathscr{Q}, \mathscr{T} : X^{s,r} \mapsto X^{s,r}$ are bounded and their norms depend only on Ω, s, r, and λ.

Theorem 4.2. *Let $\lambda \geq 0$, $s \geq 0$ and $r \in (1,\infty)$. Then there exist operators $\mathscr{F}, \mathscr{G} : H^{s,r}(\Omega) \to H^{s+1,r}$, satisfying the inequality*

$$\|\mathscr{F}\varphi\|_{H^{s+1,r}(\Omega,\lambda)} + \|\mathscr{G}\varphi\|_{H^{s+1,r}(\Omega)} \leq c(\Omega, s, r, \lambda)\|\varphi\|_{s,r} \quad \forall\varphi \in H^{s,r}(\Omega), \tag{4.2}$$

such that the representation

$$\operatorname{div}\mathbf{v} = \sigma(\mathbf{I} - (\lambda+2)^{-1}\mathscr{Q})\varphi + \sigma\mathscr{F}\varphi + (\mathscr{T} + \mathscr{G})\operatorname{div}\Delta^{-1}\mathbf{P} \tag{4.3}$$

holds true for any solution $\mathbf{v} \in H^{s+1,r}(\Omega)$ to boundary value problem (1.19).

Proof. Applying to both sides of (4.1) the operator $\operatorname{div}\Delta^{-1}$ we arrive to the following equation for $\operatorname{div}\mathbf{v}$,

$$(\mathbf{I} + \lambda\mathscr{A})(\operatorname{div}\mathbf{v}) = F, \text{ where } F = \omega\mathscr{A}\varphi + \operatorname{div}\Delta^{-1}\mathbf{P}, \quad \mathscr{A} = \operatorname{div}\Delta^{-1}\nabla \tag{4.4}$$

which, by virtue of Theorem 1.9, can be rewritten in the equivalent form

$$\Big((1+\lambda)\mathbf{I} - \frac{\lambda}{2}\mathscr{Q} + \lambda\mathscr{K}\Big)(\operatorname{div}\mathbf{v}) = F.$$

Applying to both sides of this equation the operator $\mathscr{T}$ and using the identity $\mathscr{Q}^2 = \mathscr{Q}$ we obtain

$$\big(\mathbf{I} + \lambda\mathscr{T}\mathscr{K}\big)(\operatorname{div}\mathbf{v}) = \mathscr{T}F. \tag{4.5}$$

By Theorem 1.9 the operator $\mathscr{K} : H^{s,r}(\Omega) \to H^{1+s,r}(\Omega)$ is bounded, and hence the operator $\mathscr{T}\mathscr{K} : H^{s,r}(\Omega) \to H^{s,r}(\Omega)$ is compact. It follows from this that for $\lambda \neq -1, -2$, operator equation (4.4) is a Fredholm equation. Let us prove the uniqueness of solutions to this equation. Let $v \in H^{s,r}(\Omega)$ satisfies the equation $(\mathbf{I} + \lambda\mathscr{T}\mathscr{K})v = 0$. It follows from the compactness of $\mathscr{K}$ and bootstrap arguments that v belongs $H^{s,t}(\Omega)$ for all $t > 0$ and, in particular, $v \in L^2(\Omega)$. Since $\mathscr{A}$ is nonnegative in $L^2(\Omega)$, we conclude from (4.4) that $v = 0$. Hence a solution is unique, which along with the Fredholm Theorem implies the existence of a bounded inverse $(\mathbf{I}+\lambda\mathscr{T}\mathscr{K})^{-1} : H^{s,r}(\Omega)\mapsto H^{s,r}(\Omega)$. It follows from this that

$$\operatorname{div}\mathbf{v} = (\mathscr{T}+\mathscr{G})F, \tag{4.6}$$

where

$$\mathscr{G} = -\lambda\mathscr{T}\mathscr{K}(\mathbf{I}+\lambda\mathscr{T}\mathscr{K})^{-1}\mathscr{T} : H^{s,r}(\Omega)\mapsto H^{s+1,r}(\Omega)$$

is the bounded linear operator with the norm depending only on Ω, s, r, and λ. Next note that $F = \omega(\mathbf{I}-2^{-1}\mathscr{Q}+\mathscr{K})\varphi + \operatorname{div}\Delta^{-1}\mathbf{P}$. Substituting this identity into (4.6) we obtain representation (4.3) with the operator $\mathscr{F} = (\lambda+1)(\mathscr{T}\mathscr{K}+\mathscr{G}\mathscr{A})$, which completes the proof. □

Corollary 4.3. *Assume that a vector function* $\mathbf{P}$ *admits representation* $\mathbf{P} = \operatorname{div}\mathbb{P}+\mathbf{p}$. *Then a solution to problem* (1.19) *satisfies the inequalities*

$$\|\mathbf{v}\|_{H^{1,t}} \le c\big(\omega\|\varphi\|_{L^t(\Omega)} + |\mathbf{P}|_t\big) \text{ for all } t\in(1,\infty), \tag{4.7}$$

$$\|\mathbf{v}\|_{1+s,r} \le c\big(\omega\|\varphi\|_{s,r} + |\mathbf{P}|_{s,r}\big) \text{ for all } r\in(1,\infty),\ s\in(0,1), \tag{4.8}$$

where the constant c *depends only on* Ω, λ, *and exponents* s, r, t.

Proof. Using identity (4.3) we can rewrite equation (1.19) in the form

$$\mathbf{v} = \Delta^{-1}\nabla\big(\omega\varphi - \lambda\sigma(\mathbf{I}-(\lambda+2)^{-1}\mathscr{Q})\varphi - \lambda\sigma\mathscr{F}\varphi - \lambda(\mathscr{T}+\mathscr{G})\operatorname{div}\Delta^{-1}\mathbf{F}\big) + \Delta^{-1}\mathbf{P}$$

Since

$$\|\operatorname{div}\Delta^{-1}\mathbf{P}\|_{L^t(\Omega)} \le c|\mathbf{P}|_t, \quad \|\operatorname{div}\Delta^{-1}\mathbf{P}\|_{s,r} \le c|\mathbf{P}|_{s,r},$$

estimates (4.7), (4.8) obviously follows from Theorem 4.2 and Lemma 4.1. □

Corollary 4.4. *Assume that* s, r *and* $\mathbf{U}$ *comply with conditions of Theorem 3.1, and* $(\mathbf{v},\varphi,\vartheta)\in(X^{1+s,r})^3\times X^{s,r}\times X^{1+s,r}$ *is a solution to problem* (2.3). *Then the function* φ *satisfies the operator transport equation*

$$\mathbf{u}\cdot\nabla\varphi + \sigma(1-(\lambda+2)^{-1}\mathscr{Q})\varphi + \sigma\mathscr{F}\overline{\varphi} = (\lambda+1)(\lambda+2)^{-1}m + Q_0 + \operatorname{div}(\vartheta u), \tag{4.9}$$

where

$$Q_0 = G - (\mathscr{T}+\mathscr{G})\operatorname{div}\Delta^{-1}(k\mathscr{U}\mathbf{v}+\mathbf{F}). \tag{4.10}$$

Recall that $\overline{\varphi} = \Pi\varphi$, $\quad m = (\mathbf{I}-\Pi)\varphi$.

Proof. We begin with the observation that for $\mathbf{F}=0$ and $\varphi=1$, zero is the only solution to problem (1.19), which along with (4.3) yields the identity $\big((\mathbf{I}-(\lambda+2)^{-1}\mathscr{Q})+\sigma\mathscr{F}\big)(\mathbf{I}-\Pi)=0$. Next notice that a solution to problem (2.3) satisfies equations (4.1) with $\mathbf{F}$ replaced with $k\mathscr{U}\mathbf{v}+\mathbf{F}$. It follows from this that for such a solution identity (4.3) can be written in the form

$$\operatorname{div}\mathbf{v} = \sigma\Big((\mathbf{I}-(\lambda+2)^{-1}\mathscr{Q})+\mathscr{F}\Big)\Pi\varphi + (\mathscr{T}+\mathscr{G})\operatorname{div}\Delta^{-1}(k\mathscr{U}\mathbf{v}+\mathbf{F}).$$

Substituting this equality into equation (2.3b) and recalling the relations $\varphi = m+\overline{\varphi}$, $\overline{\varphi}=\Pi\varphi$, we obtain (4.9) and the assertion follows. □

5. Model linear problem. Existence of strong solution

Using Corollary 4.4 we can replace mass balance equation (2.3b) with transport equation (4.9), and thereby split (2.3) into elliptic and hyperbolic parts. The principal parts of the modified problem reads

$$\begin{aligned} \mathbf{u}\cdot\nabla\varphi+\sigma(\mathbf{I}-\frac{1}{\lambda+2}\mathscr{Q})\varphi &= Q+\operatorname{div}(\vartheta\mathbf{u}) \text{ in } \Omega, \\ \Delta\vartheta-k\nabla\vartheta+kb\operatorname{div}(\mathbf{u}\overline{\varphi}) &= S \text{ in } \Omega \\ \varphi=0 \text{ on } \Sigma_{\text{in}}, \quad \vartheta &= 0 \text{ on } \partial\Omega. \end{aligned} \tag{5.1a}$$

$$\Delta\mathbf{v}+\lambda\nabla\operatorname{div}\mathbf{v}-\omega\nabla\varphi=\mathbf{P} \text{ in } \Omega, \quad \mathbf{v}=0 \text{ on } \partial\Omega. \tag{5.1b}$$

In this section we prove the existence and uniqueness of strong solutions to problem (5.1). We shall assume that given functions $\mathbf{P}=\operatorname{div}\mathbb{P}+\mathbf{p}$, $S=\operatorname{div}\mathbf{s}+s_0$, and Q have the finite norms $|\mathbf{P}|_{s,r}$, $|S|_{s,r}$ and $\|Q\|_{s,r}$. Next introduce the quantities

$$\mathfrak{E}_{s,r}=\|Q\|_{s,r}+|\mathbf{P}|_{s,r}+|S|_{s,r}, \quad \mathfrak{E}_t=\|Q\|_{L^t(\Omega)}+|\mathbf{P}|_t+|S|_t, \tag{5.2}$$

where the norms $|\cdot|_t$ and $|\cdot|_{s,r}$ are defined by formulas (2.5), (2.6), respectively.

Theorem 5.1. *Let exponents s,r, and a vector field $\mathbf{u}\in X^{1+s,r}$ with $\|\mathbf{u}\|_{s,r}\le R$ meet all requirements of Theorems 1.7 and 1.8, and C be the constant from Theorem 1.8. Furthermore, assume that the viscosity ratio λ satisfy the inequality* (1.28). *Then there are constants ω^*, k^*, and c, depending only on s,r,λ, R, and Ω, such that for all $\omega>\omega^*$, $k\in[0,k^*)$, problem* (2.3) *has a unique solution satisfying the inequalities*

$$\sigma\|\varphi\|_{L^t(\Omega)}+\|\mathbf{v}\|_{H^{1,t}(\Omega)}\le c\mathfrak{E}_t, \quad \sigma\|\varphi\|_{s,r}+\|\mathbf{v}\|_{1+s,r}\le c\mathfrak{E}_{s,r}+\sigma^{2\kappa}\mathfrak{E}_r, \tag{5.3a}$$

$$\|\vartheta\|_{1+s,r}\le c\sigma^{-1}\,\mathfrak{E}_{s,r}+c\sigma^{-1+2\kappa}\mathfrak{E}_r+c|S|_{s,r}. \tag{5.3b}$$

Proof. The proof is based on the following consequence of Lemma 4.1.

Lemma 5.2. *Under the assumptions of Theorem 5.1, there exist $k^*>0$ and $c>0$ depending only on Ω, r,s and $\|\mathbf{u}\|_{1+s,r}$, such that for all $H=\operatorname{div}\mathbf{h}+h_0$ and $0\le k\le k^*$, the following problem*

$$\Delta\vartheta-k\mathbf{u}\cdot\nabla\vartheta=H \text{ in } \Omega, \quad \vartheta=0 \text{ on } \partial\Omega,$$

has a unique solution satisfying the inequality $\|\vartheta\|_{1+s,r}\le c|H|_{s,r}$. Moreover, $\|\vartheta\|_{H^{1,t}(\Omega)}\le c|H|_t$ for all $t\in(1,r]$.

Let us return to the proof of Theorem 5.1. Since equations (5.1a) and (5.1b) are independent, we begin the proof with the following construction of solutions to boundary value problem (5.1a). By virtue of Theorems 1.7, 1.9, and Lemma 5.2, the recurrent system of equations

$$\begin{aligned} \mathbf{u}\cdot\nabla\varphi_n+\sigma\varphi_n=\sigma(\lambda+2)^{-1}\mathscr{Q}\varphi_{n-1}+\operatorname{div}(\vartheta_n\mathbf{u})+Q \text{ in } \Omega, \quad \varphi=0 \text{ on } \Sigma_{\text{in}}, \\ \Delta\vartheta_n-k\mathbf{u}\cdot\nabla\vartheta_n+kb\operatorname{div}(\mathbf{u}\overline{\varphi}_{n-1})=S, \end{aligned} \tag{5.4}$$

$\varphi_0=\vartheta_0=0$, has a sequences of functions $\varphi_n\in X^{s,r}$, $\vartheta_n\in X^{1+s,r}$. Moreover, Fichera-Oleinik estimate (1.9) implies the inequalities

$$\|\varphi_{n+1}-\varphi_n\|_{L^t(\Omega)}\le\big(\sigma-\|\operatorname{div}\mathbf{v}\|_{C(\Omega)}\big)^{-1}\big(\sigma(\lambda+2)^{-1}|||\mathscr{Q}|||_t\|\varphi_n-\varphi_{n-1}\|_{L^t(\Omega)}+c\|\vartheta_{n+1}-\vartheta_n\|_{H^{1,t}(\Omega)}\big).$$

On the other hand, Lemma 5.2 yields the estimate

$$\|\vartheta_{n+1}-\vartheta_n\|_{H^{1,t}(\Omega)}\le c\|\varphi_n-\varphi_{n-1}\|_{L^t(\Omega)}$$

which leads to

$$\|\varphi_{n+1}-\varphi_n\|_{L^t(\Omega)} \le \sigma^{-1}\big(\big(\sigma-\|\operatorname{div}\mathbf{v}\|_{C(\Omega)}\big)^{-1}\big((\lambda+2)^{-1}|||\mathscr{Q}|||_t+c)\|\varphi_n-\varphi_{n-1}\|_{L^t(\Omega)}\big)\ .$$

It follows from this and condition (1.28) that the estimates

$$\|\varphi_{n+1}-\varphi_n\|_{L^t(\Omega)} \le q\|\varphi_n-\varphi_{n-1}\|_{L^t(\Omega)} \quad , q<1,$$

hold true for all sufficiently large σ and $t \in [6, r]$. Hence there exists σ^*, depending only on λ and $\|\operatorname{div}\mathbf{u}\|_{C(\Omega)}$, such that for all $\sigma > \sigma^*$, the sequence φ_n converges in $L^t(\Omega)$ to a function φ and the sequence ϑ_n converges in $H^{1,t}(\Omega)$ to a function ϑ. Next, (1.9) implies

$$\|\varphi_n\|_{L^t(\Omega)} \le c\big(\sigma-\|\operatorname{div}\mathbf{v}\|_{C(\Omega)}\big)^{-1}\big(\sigma(\lambda+2)^{-1}|||\mathscr{Q}|||_t\|\varphi_{n-1}\|_{L^t(\Omega)}+c\|\vartheta_n\|_{H^{1,t}(\Omega)}+\|Q\|_{L^t(\Omega)}\big).$$

On the other hand, Lemma 5.2 yields the estimate

$$\|\vartheta_n\|_{H^{1,t}(\Omega)} \le c\|\varphi_{n-1}\|_{L^t(\Omega)} + c|S|_t\ .$$

Combining the obtained estimates and recalling (1.28) we finally obtain that for all large σ,

$$\|\varphi_n\|_{L^t(\Omega)} \le q\|\varphi_{n-1}\|_{L^t(\Omega)} + c\sigma^{-1}(\|Q\|_{L^t(\Omega)} + |S|_t), \quad 0<q<1,$$

which along with Lemma 5.2 gives the estimate

$$\|\varphi\|_{L^t(\Omega)} + \|\vartheta\|_{H^{1,t}(\Omega)} \le c\sigma^{-1}(\|Q\|_{L^t(\Omega)} + |S|_t), \quad 6 \le t \le r. \tag{5.5}$$

Applying estimate (1.17) in Theorem 1.7 to equation (5.1a) we obtain

$$\begin{aligned}\|\varphi_n\|_{s,r} \le{}& C(\lambda+2)^{-1}\big(|||\mathscr{Q}|||_{s,r}\|\varphi_{n-1}\|_{s,r} + \sigma^{2\kappa}|||\mathscr{Q}|||_r\|\varphi_n\|_{L^r(\Omega)}\big)+\\ & C\sigma^{-1}(\|\vartheta_n\|_{1+s,r} + \|Q\|_{s,r}) + +C\sigma^{-1+2\kappa}(\|\vartheta_n\|_{H^{1,r}(\Omega)} + \|Q\|_{L^r(\Omega)}).\end{aligned}$$

On the other hand, Lemma 5.2 and (5.4) imply the inequality

$$\|\vartheta_n\|_{1+s,r} \le c\|\varphi_{n-1}\|_{s,r} + |S|_{s,r.} \tag{5.6}$$

From this and (1.28) we derive the inequalities

$$\begin{aligned}\|\varphi_n\|_{s,r} \le{}& q\|\varphi_{n-1}\|_{s,r} + \sigma^{2\kappa}C\|\varphi_{n-1}\|_{L^r(\Omega)} + c\sigma^{-1}(\|Q\|_{s,r} + |H|_{s,r})+\\ & c\sigma^{-1+2\kappa}(\|Q\|_{L^r(\Omega)} + |H|_r), \quad q<1.\end{aligned}$$

Combining this result with (5.5) we finally obtain

$$\|\varphi_n\|_{s,r} \le c\sigma^{-1}(\|Q\|_{s,r} + |S|_{s,r}) + c\sigma^{-1+2\kappa}(\|Q\|_{L^r(\Omega)} + |H|_r).$$

Hence the sequence φ_n is bounded and converges weakly in $X^{s,r}$ to the function $\varphi \in X^{s,r}$ satisfying (5.3a). Estimate (5.3b) for ϑ easy follows from the estimate for φ and (5.6). It remains to note that estimates for $\mathbf{v}$ follow from the estimates for φ and inequalities (4.7)-(4.8) in Corollary (4.3). $\square$

6. Linearized equations. Strong solutions

We are now in a position to prove the main theorem on solvability of basic problem (2.3).

Theorem 6.1. *Let exponents s, r, a vector field $\mathbf{u} \in X^{1+s,r}$, and the viscosity ratio λ meet al requirements of Theorem 5.1, and*

$$\|\mathbf{u}\|_{1+s,r} + \|\mathbf{a}\|_{1+s,r} \le R,$$

where $\mathbf{a}$ is the vector field given in the definition of operator $\mathscr{U}$. Furthermore, assume that $G \in X^{s,r}$, the functions $\mathbf{F}$ and H have the finite norms $|\mathbf{F}|_{s,r}$ and $|H|_{s,r}$. Then there are constants ω^*, k^* and c depending only on R, s, r, λ, and Ω, such that for all $\omega > \omega^*$, $k \in [0, k^*)$, problem (2.3) has a unique solution which admits the estimate

$$\omega\|\varphi\|_{s,r} + \|\mathbf{v}\|_{1+s,r} \le c\omega^{2\kappa}\big((\omega\|\overline{\varphi}\|_{L^2(\Omega)} + \omega|m| + \|\mathbf{v}\|_{H^{1,2}(\Omega)})\big) + (|\mathbf{F}|_{s,r} + |H|_{s,r} + \|G\|_{s,r}), \tag{6.1a}$$

$$\|\vartheta\|_{1+s,r} \le c\|\varphi\|_{s,r} + c\omega^{-1}\|\mathbf{v}\|_{1+s,r} + c|H|_{s,r}. \tag{6.1b}$$

Proof. Applying Corollary 4.4 we can replace the mass balance equation (2.3b) in (2.3) by transport equation (4.9) and rewrite the system (2.3) in the equivalent form

$$\begin{aligned} \Delta\mathbf{v} + \lambda\nabla\operatorname{div}\mathbf{v} - \omega\nabla\varphi = \mathbf{P} \text{ in } \Omega,\\ \mathbf{u}\cdot\nabla\varphi + \sigma(\mathbf{I} - \frac{1}{\lambda+2}\mathscr{Q})\varphi - \operatorname{div}(\vartheta\mathbf{u}) = -\sigma\mathscr{F}\overline{\varphi} + \sigma\frac{\lambda+1}{\lambda+2}m + Q_0 \text{ in } \Omega,\\ \Delta\vartheta - k\nabla\vartheta + kb\operatorname{div}(\mathbf{u}\overline{\varphi}) = S \text{ in } \Omega\\ \varphi = 0 \text{ on } \Sigma_{\rm in}, \quad \vartheta = 0, \quad \mathbf{v} = 0 \text{ on } \partial\Omega. \end{aligned} \tag{6.2}$$

where

$$\begin{aligned} \mathbf{P} \equiv k\mathscr{U}\mathbf{v} + \mathbf{F} =: \operatorname{div}\mathbb{P} + \mathbf{p}, \quad S \equiv \frac{k}{\omega}\mathscr{W}\mathbf{v} + H =: \operatorname{div}\mathbf{s} + s_0,\\ \mathbb{P} = k\mathbf{a}\otimes\mathbf{v} + k\mathbf{v}\otimes\mathbf{a} + \mathbb{F}, \quad \mathbf{p} = \mathbf{f},\\ \mathbf{s} = -\frac{2kb}{\omega}(\nabla\mathbf{u}_0 + \nabla\mathbf{u}_0^*)\mathbf{v} + \mathbf{h}, \quad s_0 = \frac{2kb}{\omega}\Delta\mathbf{u}_0\cdot\mathbf{v} + h_0. \end{aligned} \tag{6.3}$$

$$Q_0 = G - (\mathscr{T} + \mathscr{G})\operatorname{div}\Delta^{-1}\mathbf{P}. \tag{6.4}$$

Each solution to problem (6.2) can be formally regarded as a solution to equations (5.1) with

$$Q = \sigma(\lambda+1)(\lambda+2)^{-1}m + \sigma\mathscr{F}\varphi + Q_0. \tag{6.5}$$

Thus, we can derive a priori estimates for solutions to problem (6.2) by using inequalities (5.3). We begin with the estimates

$$\begin{aligned} |\mathbf{P}|_{s,r} = \|\mathbb{P}\|_{s,r} + \|\mathbf{p}\|_{s,r} \le ck\|\mathbf{v}\|_{s,r} + |\mathbf{F}|_{s,r} \le c\|\mathbf{v}\|_{H^{1,r}(\Omega)} + |\mathbf{F}|_{s,r},\\ |S|_{s,r} = \|\mathbf{s}\|_{s,r} + \|s_0\|_{s,r} \le ck\omega^{-1}\|\mathbf{v}\|_{s,r} + |H|_{s,r} \le ck\omega^{-1}\|\mathbf{v}\|_{H^{1,r}(\Omega)} + |H|_{s,r}, \end{aligned} \tag{6.6}$$

¿From this and Lemma 4.1 we conclude that

$$\|\operatorname{div}\Delta^{-1}\mathbf{P}\|_{s,r} \le c\|\Delta^{-1}\mathbf{P}\|_{1+s,r} \le c|\mathbf{P}|_{s,r} \le c(k\|\mathbf{v}\|_{H^{1,r}(\Omega)} + |\mathbf{F}|_{s,r}),$$

which, along with estimate (4.2) for the operator $\mathscr{G}$, and expression (6.4) for Q_0, implies the inequality

$$\|Q_0\|_{s,r} \le c(k\|\mathbf{v}\|_{H^{1,r}(\Omega)} + |\mathbf{F}|_{s,r}).$$

On the other hand, it follows from Theorem 4.2 that

$$\|\mathscr{F}\overline{\varphi}\|_{s,r} \le c\|\mathscr{F}\overline{\varphi}\|_{H^{1,r}(\Omega)} \le c\|\overline{\varphi}\|_{L^r(\Omega)}.$$

Combining these inequalities with expression (6.5) for Q we obtain the estimate

$$\|Q\|_{s,r} \le c(\|G\|_{s,r} + |\mathbf{F}|_{s,r}) + \sigma|m| + \sigma\|\overline{\varphi}\|_{L^r(\Omega)} + \|\mathbf{v}\|_{H^{1,r}(\Omega)} \ .$$

Introduce the quantities

$$\begin{aligned}\mathfrak{F}_{s,r} = \|G\|_{s,r} + |\mathbf{F}|_{s,r} + |H|_{s,r}, \quad \mathfrak{F}_t = \sigma\|G\|_{L^t(\Omega)} + |\mathbf{F}|_t + |H|_t,\\ \mathfrak{G}_{s,r} = \sigma\|\varphi\|_{s,r} + \|\mathbf{v}\|_{s,r}, \quad \mathfrak{G}_t = \sigma\|\varphi\|_{L^t(\Omega)} + \|\mathbf{v}\|_{H^{1,t}(\Omega)}.\end{aligned} \tag{6.7}$$

With this notation we have

$$\|Q\|_{s,r} \le c\mathfrak{F}_{s,r} + c\mathfrak{G}_r + \sigma|m|,$$

which along with (6.6) gives

$$\mathfrak{E}_{s,r} \equiv (|\mathbf{P}|_{s,r} + |S|_{s,r} + \|Q\|_{s,r}) \le c|\mathfrak{F}|_{s,r} + c\mathfrak{G}_r + c\sigma|m|.$$

From this and inequality (5.3a) we obtain

$$\mathfrak{G}_{s,r} \le c\mathfrak{F}_{s,r} + c\mathfrak{G}_r + c\sigma|m| + \sigma^{2\kappa}\mathfrak{E}_r \tag{6.8}$$

Now our task is to estimate $\mathfrak{E}_r$ and $\mathfrak{G}_r$ Repeating the previous arguments we arrive at the inequalities

$$|\mathbf{P}|_r = \|\mathbb{P}\|_{L^r(\Omega)} + \|\mathbf{p}\|_{L^r(\Omega)} \le ck\|\mathbf{v}\|_{L^r(\Omega)} + |\mathbf{F}|_r \le c\|\mathbf{v}\|_{H^{1,6}(\Omega)} + |\mathbf{F}|_r,$$

$$|S|_r = \|\mathbf{s}\|_{L^r(\Omega)} + \|s_0\|_{L^r(\Omega)} \le ck\omega^{-1}\|\mathbf{v}\|_{L^r(\Omega)} + |H|_r \le ck\omega^{-1}\|\mathbf{v}\|_{H^{1,6}(\Omega)} + |H|_r.$$

Thus, we get

$$\|Q_0\|_r \le c(k\|\mathbf{v}\|_{H^{1,6}(\Omega)} + |\mathbf{F}|_r).$$

and, by virtue of Theorem 4.2, that

$$\|\mathscr{F}\overline{\varphi}\|_r \le c\|\mathscr{F}\overline{\varphi}\|_{H^{1,6}(\Omega)} \le c\|\overline{\varphi}\|_{L^6(\Omega)}.$$

Here we use the continuity of embedding $L^r(\Omega) \hookrightarrow H^{1,6}(\Omega)$. Combining these estimates with the expressions for Q_0 and Q we obtain

$$\mathfrak{E}_r \le c\mathfrak{F}_r + c\mathfrak{G}_6 + \sigma|m|. \tag{6.9}$$

Substituting this result in inequality (5.3a) with $t = r$ we arrive at the estimate

$$\mathfrak{G}_r \le c\mathfrak{F}_r + c\mathfrak{G}_6 + c\sigma|m|. \tag{6.10}$$

It remains to estimate $\mathfrak{G}_6$. We have

$$|\mathbf{P}|_6 = \|\mathbb{P}\|_{L^6(\Omega)} + \|\mathbf{p}\|_{L^6(\Omega)} \le ck\|\mathbf{v}\|_{L^6(\Omega)} + |\mathbf{F}|_6 \le c\|\mathbf{v}\|_{H^{1,2}(\Omega)} + |\mathbf{F}|_6,$$

$$|S|_6 = \|\mathbf{s}\|_{L^6(\Omega)} + \|s_0\|_{L^6(\Omega)} \le ck\omega^{-1}\|\mathbf{v}\|_{L^6(\Omega)} + |H|_6 \le ck\omega^{-1}\|\mathbf{v}\|_{H^{1,2}(\Omega)} + |H|_6.$$

$$\|\mathscr{F}\overline{\varphi}\|_{L^6(\Omega)} \le c\|\mathscr{F}\overline{\varphi}\|_{H^{1,2}(\Omega)} \le c\|\overline{\varphi}\|_{L^2(\Omega)}.$$

Thus, we get the inequality $\mathfrak{E}_6 \le c\mathfrak{F}_6 + c\mathfrak{G}_2 + \sigma|m|$, which together with (5.3a) gives the estimate

$$\mathfrak{G}_6 \le c\mathfrak{F}_6 + c\mathfrak{G}_2 + c\sigma|m|. \tag{6.11}$$

Substituting (6.11) into (6.10) and then into (6.8) we finally obtain

$$\mathfrak{G}_{s,r} \le c\mathfrak{F}_{s,r} + c\sigma^{2\kappa}(\mathfrak{F}_r + \mathfrak{F}_6 + \mathfrak{G}_2 + \sigma|m|) \le c\sigma^{2\kappa}(\mathfrak{F}_{s,r} + \mathfrak{G}_2 + \sigma|m|).$$

Since $\sigma = (\lambda+1)^{-1}\omega$, this estimate implies desired inequality (6.1a). Estimate (6.1b) for ϑ easily follows from (6.1a) and Lemma 5.2. In particular, a solution $(\mathbf{v}, \varphi, \vartheta) \in (X^{1+s,r})^3 \times X^{s,r} \times X^{1+s,r}$ is unique.

Since the equations (6.2) differ from equations (5.1) by lower-order compact terms, the existence of a solution to (6.2) follows from its uniqueness and the Fredholm alternative. □

At the end of the section we touch on the adjoint problem. Direct calculations show that the formal adjoint of equations and boundary conditions (2.3) reads

$$\begin{aligned} \Delta \mathbf{w} + \lambda \nabla \operatorname{div} \mathbf{w} - \nabla \psi - k\mathscr{U}^*\mathbf{w} - k\omega^{-1}\mathscr{W}^*\chi = \mathbf{F}^* \text{ in } \Omega, \\ -\operatorname{div}(\mathbf{u}\cdot\nabla\varphi) + \omega \operatorname{div}\mathbf{w} - k\Pi(\mathbf{u}\nabla\chi) = G^* \text{ in } \Omega, \\ \Delta\chi + k\operatorname{div}(\chi\mathbf{u}) + \operatorname{div}(\mathbf{u}\Pi\psi) - k\operatorname{div}\mathbf{u}\Pi\psi = H^*, \text{ in } \Omega \\ \psi = 0 \text{ on } \Sigma_{\text{out}}, \quad \mathbf{w} = 0, \quad \chi = 0 \text{ on } \partial\Omega. \end{aligned} \tag{6.12}$$

The analysis of problem (6.12) is similar to that of problem (2.3). Further, we shall use the following particular result.

Theorem 6.2. *Under the assumptions of Theorem 6.1, there are constants ω^*, k^* and c, depending only on R, s, r, λ, and Ω, such that for all $\omega > \omega^*$, $k \in [0, k^*)$, problem* (6.12) *has a unique solution $(\mathbf{w}, \psi, \chi) \in (X^{1+s,r})^3 \times X^{s,r} \times X^{1+s,r}$ which satisfies the following estimates*

$$\begin{aligned} \|\overline{\psi}\|_{L^2(\Omega)} \le c|\mathbf{F}^*|_2 + c\omega^{-1/2+\varepsilon}(\|G^*\|_{L^2(\Omega)} + |H^*|_2), \\ \|\chi\|_{H^{1,2}(\Omega)} \le c|\mathbf{F}^*|_2 + c\omega^{-1/2+\varepsilon}\|G^*\|_{L^2(\Omega)} + c|H^*|_2. \end{aligned} \tag{6.13}$$

Proof. The change of variable $\mathbf{w}^* = \omega\mathbf{w}$ brings the equations to the form

$$\begin{aligned} \Delta \mathbf{w}^* + \lambda \nabla \operatorname{div} \mathbf{w}^* - \omega\nabla \psi - k\mathscr{U}^*\mathbf{w}^* - k\mathscr{W}^*\chi = \omega\mathbf{F}^* \text{ in } \Omega, \\ -\operatorname{div}(\mathbf{u}\cdot\nabla\varphi) + \operatorname{div}\mathbf{w}^* - k\Pi(\mathbf{u}\nabla\chi) = G^* \text{ in } \Omega, \\ \Delta\chi + k\operatorname{div}(\chi\mathbf{u}) - \operatorname{div}(\mathbf{u}\Pi\psi) + \operatorname{div}\mathbf{u}\Pi\psi = H^*, \text{ in } \Omega \\ \psi = 0 \text{ on } \Sigma_{\text{out}}, \quad \mathbf{w}^* = 0, \quad \chi = 0 \text{ on } \partial\Omega. \end{aligned} \tag{6.14}$$

Equations (6.14) are similar to equations (2.3), and the proof of Theorem 6.1 remains valid for Theorem 6.2. The only remark is that the references to Theorem 1.7 should be replaced by those to Theorem 1.8. Estimates (6.13) follow from energy estimate (3.1b), which holds true for solutions to problem (6.14). □

7. Boundary layer approximation

In this section we briefly discuss a boundary layer phenomena. The formal application of Theorem 6.1 to problem (1.1) shows that $\|\varphi\|_{s,r} \sim \omega^{-1+2\kappa+\varepsilon}$ and $\|\mathbf{v}\|_{1+s,r} \sim \omega^{2\kappa+\varepsilon}$ for large ω. In other words, $\nabla\mathbf{v}$ develops singularity at the inlet as $\omega \to \infty$. From the mathematical standpoint, such a behavior is caused by two factors: the first is the accretivity defect $\kappa \ne 0$, and the second is a disparity between the density mean value $(\mathbf{I} - \Pi)\varphi$ and the deviation $\Pi\varphi$. Or aim is to show that in the case of "well prepared data" satisfying condition ($\mathbf{H}\,2$), problem (1.1) has a regular solution for all sufficiently large ω.

To this end notice that the right- hand side of governing equations (2.2) can be split into two parts: the leading part $(0, \Upsilon_1, \Theta_1)$ of order ω^{-1}, and small quadratic terms $(\Psi, \Upsilon_2, \Theta_2)$. In this section we deduce estimates for solutions to the boundary value problem

$$\begin{aligned}\Delta \mathbf{v}_1 + \lambda \nabla \operatorname{div} \mathbf{v}_1 - \omega \nabla \varphi_1 &= \mathscr{U} \mathbf{v}_1 \text{ in } \Omega,\\ \mathbf{u} \cdot \nabla \varphi_1 + \operatorname{div} \mathbf{v}_1 - \operatorname{div}(\mathbf{u}\vartheta_1) &= \Upsilon_1 \text{ in } \Omega,\\ \Delta \vartheta_1 - k\nabla \vartheta_1 + kb \operatorname{div}(\mathbf{u}\overline{\varphi}_1) &= k\omega^{-1}\mathscr{W}\mathbf{v}_1 + \Theta_1, \text{ in } \Omega\\ \varphi_1 = 0 \text{ on } \Sigma_{\text{in}}, \quad \mathbf{v}_1 = 0, &\quad \vartheta_1 = 0 \text{ on } \partial\Omega,\end{aligned} \tag{7.1}$$

which can be regarded as the principal part of problem (1.1). Recall that functions Υ_1 and Θ_1 are defined by formulae (2.2f). The following theorem is the main result of this section.

Theorem 7.1. *Let exponents s, r, a vector field $\mathbf{u} \in X^{1+s,r}$, the viscosity ratio λ and constants ω^*, k^* comply with hypotheses of Theorem 6.1, and $\varepsilon > 0$. Then there is a constant c depending only on R, $s, r, \lambda, \varepsilon$, and Ω such that for all $\omega > \omega^*$, $k \in [0, k^*)$, problem (7.1) has a unique solution which admits the estimates*

$$\omega \|\varphi_1\|_{s,r} + \|\mathbf{v}_1\|_{1+s,r} \le c\omega^{-1/2+2\kappa+\varepsilon}, \quad \|\vartheta_1\|_{1+s,r} \le c\omega^{-1} + c\omega^{-3/2+2\kappa+\varepsilon}. \tag{7.2}$$

Proof. It follows from the expression (2.2f) that functions Υ_1 and Θ_1 have the bounds

$$|\Upsilon_1| + |\Theta_1| \le c(\|\Upsilon_1\|_{s,r} + \|\Theta_1\|_{s,r}) \le c\omega^{-1}, \tag{7.3}$$

where the constant c depends only on $\|\mathbf{u}\|_{1+s,r}$, Ω, s, r , and p_0, Φ_0. Equations (7.1) meet all requirements of Theorem 6.1 and their solutions satisfy inequalities (6.1a) with $\mathbf{F} = 0$, $G = \Upsilon_1$, $H = \Theta_1$. In particular, estimate (6.1a), together with the energy estimates (3.1a), (3.1b), implies the inequality

$$\begin{aligned}\omega \|\varphi_1\|_{s,r} + \|\mathbf{v}_1\|_{1+s,r} &\le c\omega^{2\kappa}(\omega|m|_1 + \|\overline{\varphi}_1\|_{L^2(\Omega)} + \|\mathbf{v}_1\|_{H^{1,2}(\Omega)})\\ &+ c\omega^{-1+2\kappa} \le c\omega^{-1/2+2\kappa+\varepsilon} + c\omega|m|_1.\end{aligned} \tag{7.4}$$

It remains to estimate $m_1 = (\mathbf{I} - \Pi)\varphi_1$. Denote by $\mathbf{w}, \psi, \chi \in (X^{1+s,r})^3 \times X^{s,r} \times X^{1+s,r}$ a solution to the adjoint problem (6.12) with the right- hand sides $\mathbf{F}^* = 0$, $H^* = 0$, $G^* = 1$. Theorem 6.2 guarantees the existence and uniqueness of such solution. Moreover, inequalities (6.13) imply the estimates

$$\|\overline{\psi}\|_{L^2(\Omega)} + \|\chi\|_{H^{1,2}(\Omega)} \le c\omega^{-1/2+\varepsilon}. \tag{7.5}$$

By the definition of adjoint problem, we have

$$\begin{aligned}m_1 \equiv \frac{1}{\text{meas } \Omega} \int_\Omega \varphi \, dx = \frac{1}{\text{meas } \Omega} \int_\Omega (\Upsilon_1 \psi + \Theta_1 \chi) \, dx =\\ \frac{1}{\text{meas } \Omega} \int_\Omega (\Upsilon_1 \overline{\psi} + \Theta_1 \chi) \, dx + (\mathbf{I} - \Pi)\psi \frac{1}{\text{meas } \Omega} \int_\Omega \Upsilon_1 \, dx\end{aligned} \tag{7.6}$$

It follows from the expression (2.2f) and condition (**H**2), that

$$\int_\Omega \Upsilon_1 \, dx = -\omega^{-1} \int_\Omega \operatorname{div}((p_0 + \Phi)\mathbf{u}) dx = -\omega^{-1} \int_\Omega (p_0 + \Phi)\mathbf{U} \cdot \mathbf{n} \, ds = 0$$

¿From this and (7.3), (7.5) we obtain

$$|m_1| \le c\|\Upsilon_1\|_{L^2(\Omega)}\|\overline{\psi}\|_{L^2(\Omega)} + c\|\Theta_1\|_{L^2(\Omega)}\|\chi\|_{L^2(\Omega)} \le c\omega^{-3/2+\varepsilon}.$$

Substituting this inequality in (7.4) gives (6.1a) and the theorem follows. □

8. Proof of Theorem 1.10

In this section we establish the local solvability of problem (2.2) and thus prove Theorem 1.10. We solve problem (2.2) by an application of the Schauder fixed point Theorem in the following framework.
Assume that Ω, $\mathbf{U}$, λ, r, s and the limiting functions $\mathbf{u}_0$, p_0 meet all requirements of theorem 1.10. Fix a positive ε such that

$$2\kappa + \varepsilon < 1/6. \tag{8.1}$$

For each $\omega > 0$, denote by $\mathfrak{M}_\omega \subset (X^{1+s,r})^3 \times X^{s,r} \times X^{1+s,r}$ a bounded convex set defined by the equalities

$$\mathfrak{M}_\omega = \big\{(\mathbf{v},\varphi,\vartheta) :\, \|\mathbf{v}\|_{1+s,r} \le \omega^{-1/3}, \quad \|\varphi\|_{s,r} \le \omega^{-4/3}, \quad \|\vartheta\|_{1+s,r} \le \omega^{-5/6}\big\}.$$

For every $(\mathbf{v},\varphi,\vartheta) \in \mathfrak{M}_\omega$, we set

$$\mathbf{u} = \mathbf{u}_0 + \mathbf{v}, \quad \mathbf{a} = \mathbf{u}_0 + 2^{-1}\mathbf{v} \tag{8.2}$$

Then there is $\omega_0(s,r)$ such that for all $\omega > \omega_0$ and $(\mathbf{v},\varphi,\vartheta) \in \mathfrak{M}_\omega$,

$$\|\mathbf{u}\|_{1+s,r} + \|\mathbf{a}\|_{1+s,r} \le R = 2\|\mathbf{u}_0\|_{1+s,r}$$

Therefore, the vector fields $\mathbf{u}$ and $\mathfrak{a}$ comply with conditions of Theorems 6.1 and 7.1. For every $(\mathbf{v},\varphi,\vartheta) \in \mathfrak{M}_\omega$ and $\mathbf{u}$, $\mathbf{a}$ defined by (8.2), consider the boundary value problem

$$\begin{aligned} \Delta\mathbf{v}_2 + \lambda\nabla\operatorname{div}\mathbf{v}_2 - \omega\nabla\varphi_2 - k\mathscr{U}\mathbf{v}_2 &= \Psi[\mathbf{v},\varphi,\vartheta], \\ \mathbf{u}\cdot\nabla\varphi_2 + \operatorname{div}\mathbf{v}_2 - \operatorname{div}(\mathbf{u}\vartheta_2) &= \Upsilon_2[\mathbf{v},\varphi,\vartheta], \\ \Delta\vartheta_2 - k\mathbf{u}\cdot\nabla\vartheta_2 + kb\operatorname{div}(\mathbf{u}\Pi\varphi_2) - k\omega^{-1}\mathscr{W}\mathbf{v}_2 &= \Theta_2[\mathbf{v},\varphi,\vartheta], \\ \mathbf{v}_2 = 0, \quad \vartheta_2 = 0 \text{ on } \partial\Omega, \quad \varphi_2 &= 0 \text{ on } \Sigma_{\text{in}}, \end{aligned} \tag{8.3}$$

where nonlinear differential operators Ψ, Υ_2, Θ_2 are defined by (2.2f). It follows from Theorem 6.1 that this problem has a unique solution $(\mathbf{v}_2,\varphi_2,\vartheta_2) \in (X^{1+s,r})^3 \times X^{s,r} \times X^{1+s,r}$. Thus, the mapping

$$\Xi : (\mathbf{v},\varphi,\vartheta) \to (\mathbf{v}_1,\varphi_1,\vartheta_1) + (\mathbf{v}_2,\varphi_2,\vartheta_2), \tag{8.4}$$

is well-defined in $\mathfrak{M}_\omega$. Recall that $(\mathbf{v}_1,\varphi_1,\vartheta_1)$ is a solution to problem (7.1) given by Theorem 7.1. Let us prove that Ξ maps $\mathfrak{M}_\omega$ into itself. Since $X^{s,r}$ and $X^{1+s,r}$ are Banach algebras, formulae (2.2f) imply the estimates

$$\begin{aligned} |\Psi|_{s,r} &\le c\big(\|\varphi|_{s,r} + \|\vartheta\|_{s,r} + \omega^{-1}\big)(1 + \omega\|\vartheta\|_{s,r}) \le c\omega^{-2/3}, \\ \|\Upsilon_2\|_{s,r} &\le c\|\varphi\|_{s,r}\|\mathbf{v}\|_{1+s,r} \le c\omega^{-4/3}, \\ \|\Theta_2\|_{s,r} &\le c\big((\|\varphi\|_{s,r} + \|\vartheta\|_{s,r} + \omega^{-1})\|\vartheta\|_{s,r} + \|\vartheta\|_{s,r} + \|\varphi\|_{s,r}\big)\|\mathbf{v}\|_{1+s,r} + \\ &\quad (\|\varphi|_{s,r} + \|\vartheta\|_{s,r} + \omega^{-1})(\|\vartheta\|_{1+s,r} + \omega^{-1}\|\mathbf{v}\|^2_{1+s,r}) \le c\omega^{-7/6}\ . \end{aligned} \tag{8.5}$$

In our notation, c denotes generic constants, which are different in different places and depend only on Ω, $\mathbf{U}$, λ, r, s, and unperturbed solution $\mathbf{u}_0$, p_0. Next, by

Theorem 3.1, a solution to problem (8.3) satisfies the energy inequalities (3.1) with $(\mathbf{F}, G, H)$ replaced by $(\Psi_2, \Upsilon_2, \Theta_2)$. It follows from this and (8.5) that

$$|m_2| + \|\overline{\varphi}_2\|_{L^2(\Omega)} + \|\mathbf{v}_2\|_{H^{1,2}(\Omega)} \leq c\omega^{-2/3+\varepsilon}. \tag{8.6}$$

Next note that equations (8.3) meet all all requirements of Theorem 6.1 and, as a consequence, functions $(\mathbf{v}_2, \varphi_2, \vartheta_2)$ satisfy inequalities (6.1) with $(\mathbf{F}, G, H)$ replaced by $(\Psi_2, \Upsilon_2, \Theta_2)$. From this and (8.5), (8.6) we get the estimates

$$\|\mathbf{v}_2\|_{1+s,r} \leq c\omega^{-2/3+2\kappa+\varepsilon}, \quad \|\varphi_2\|_{s,r} \leq c\omega^{-5/3+2\kappa+\varepsilon},$$
$$\|\vartheta_2\|_{1+s,r} \leq c\omega^{-5/3+2\kappa+\varepsilon} + \omega^{-7/6} \leq c\omega^{-7/6}.$$

Combining this result with estimates (7.2) for $(\mathbf{v}_1, \varphi_1, \vartheta_1)$ we finally obtain

$$\|\mathbf{v}_1+\mathbf{v}_2\|_{1+s,r} \leq c\omega^{-1/2+2\kappa+\varepsilon}, \quad \|\varphi_1+\varphi_2\|_{s,r} \leq c\omega^{-3/2+2\kappa+\varepsilon}, \quad \|\vartheta_1+\vartheta_2\|_{1+s,r} \leq c\omega^{-1} \tag{8.7}$$

Recall that the constant c does not depend on ω. Inequalities (8.7) and (8.1)imply the existence of a constant ω^*, depending only on Ω, $\mathbf{U}$, λ, r, s, $\mathbf{u}_0$, p_0 such that Ξ maps the set $\mathcal{M}_\omega$ into itself for all $\omega > \omega^*$.

Since the set of solutions to problems (8.3) and (7.1) is weakly compact in $(X^{1+s,r})^3 \times X^{s,r} \times X^{1+s,r}$, the mapping Ξ is weakly continuous on $\mathfrak{M}_\omega$ and, by virtue of the Schauder fixed-point theory, it has at least one fixed point $(\mathbf{v}_\omega, \varphi_\omega, \vartheta_\omega) \in \mathfrak{M}_\omega$ for all large ω. It remains to note that limiting relations (1.29) follows from the definition of $\mathfrak{M}_\omega$, which completes the proof of Theorem 1.10. □

9. Proof of Theorems 1.5-1.7

9.1. **Proof of Theorem 1.5.** Our strategy is the following. First we show that in the vicinity of each point $P \in \Sigma_{\text{in}}$ there exist normal coordinates (y_1, y_2, y_3) such that $\mathbf{u} \cdot \nabla_x = \mathbf{e}_1 \nabla_y$. Hence problem of existence of solutions to transport equation in the neighborhood of Σ_{in} is reduced to a boundary value problem for the model equation $\partial_{y_1}\varphi + \sigma\varphi = f$. Next we prove that the boundary value problem for the model equations has a unique solution in fractional Sobolev space, which leads to the existence and uniqueness of solutions in the neighborhood of the inlet set. Using the existence of local solution we reduce problem (1.7) to the problem for modified equation, which does not require the boundary data. Application of well-known results on solvability of elliptic-hyperbolic equations in the case $\Gamma = \emptyset$ gives finally the existence and uniqueness of solutions to problems (1.7).

Normal coordinates. First we introduce some notation which will be used throughout of this section. For any $a > 0$ we denote by Q_a the cube $[-a, a]^3$ and by Q_a^+ the slab $[-a, a]^2 \times [0, a]$ in the space of points $y = (y_1, y_2, y_3) \in \mathbb{R}^3$. We will write Y instead of (y_2, y_3) so that $y = (y_1, Y)$.

Definition 9.1. *A standard parabolic neighborhood associated with the constant c_0 is a compact subset of a slab Q_a^+, defined by the inequalities*

$$\mathcal{P}_a = \{y = (y_1, Y) \in Q_a^+ : a^-(Y) \leq y_1 \leq a^+(Y)\}, \tag{9.1}$$

where $a^\pm : [-a, a] \times [0, a] \mapsto \mathbb{R}$ are continuous, piece-wise C^1-functions satisfying the inequalities

$$\begin{gathered} -a \leq a^-(Y) \leq 0 \leq a^+(Y) \leq a, \\ -c_0\sqrt{y_3} \leq a^-(Y) \leq a^+(Y) \leq c_0\sqrt{y_3}, \\ |\partial_{y_2} a^\pm(Y)| \leq c_0, \quad |\partial_{y_3} a^\pm(Y)| \leq c_0/\sqrt{y_3}. \end{gathered} \tag{9.2}$$

Denote by $\Sigma^y_{\rm in}$ and $\Sigma^y_{\rm out}$ the surfaces determined by the relations

$$\Sigma^y_{\rm in} = \{y : Y \in Q_{in}, \quad y_1 = a^-(Y)\},$$
$$\Sigma^y_{\rm out} = \{y : Y \in Q_{out}, \quad y_1 = a^+(Y)\}.$$

where $Q_{in} = \{Y : a^-(Y) > -a\}$ and $Q_{out} = \{Y : a^+(Y) < a\}$. It is clear that $\partial\mathcal{P}_a = (\partial Q_a \cap \partial\mathcal{P}_a) \cup \Sigma^y_{\rm in} \cup \Sigma^y_{\rm out}$.

Lemma 9.2. *Let C^∞-manifold $\Gamma = cl\ \Sigma_{\rm in} \cup cl\ (\Sigma_{\rm out} \cap \Sigma_0)$ and a vector field $\mathbf{U} \in C^\infty(\partial\Omega)^3$ comply with Condition 1.4, and $\mathbf{u} \in C^1(\mathbb{R}^3)^3$ be a compactly supported vector field such that $\mathbf{u} = \mathbf{U}$ on $\partial\Omega$. Denote $M = \|\mathbf{u}\|_{C^1(\mathbb{R}^3)}$. Then there are positive constants a, c, C, ρ, and R, depending only on M, $\partial\Omega$, and $\mathbf{U}$, with the properties:*
($\mathbf{P}1$) *For any point $P \in \Gamma$ there exists a mapping $y \to \mathbf{x}(y)$ which takes diffeomorphically the cube Q_a onto a neighborhood $\mathcal{O}_P$ of P and satisfies the equations*

$$\partial_{y_1}\mathbf{x}(y) = \mathbf{u}(\mathbf{x}(y)) \text{ in } Q_a, \tag{9.3}$$

and the inequalities

$$\|\mathbf{x}\|_{C^1(Q_a)} + \|\mathbf{x}^{-1}\|_{C^1(\mathcal{O}_P)} \le C, \quad |\mathbf{x}(y)| \le C|y|. \tag{9.4}$$

($\mathbf{P}2$) *There is a standard parabolic neighborhood $\mathcal{P}_a$ associated with the constant c such that*

$$\mathbf{x}(\mathcal{P}_a) = \mathcal{O}_P \cup \Omega, \quad \mathbf{x}(\Sigma^y_{\rm in}) = \Sigma_{\rm in} \cap \mathcal{O}_P, \quad \mathbf{x}(\Sigma^y_{\rm out}) = \Sigma_{\rm out} \cap \mathcal{O}_P. \tag{9.5}$$

($\mathbf{P}4$) *Denote by $G_a \subset \mathcal{P}_a$ the domain*

$$G_a = \{y = (y_1, Y) \in \mathcal{P}_a : Y \in Q_{in}\}, \tag{9.6}$$

and by $B_P(\rho)$ the ball $|x - P| \le \rho$. Then we have the inclusions

$$B_P(\rho) \cap \Omega \subset \mathbf{x}(G_a) \subset \mathcal{O}_P \cap \Omega \subset B_P(R) \cap \Omega. \tag{9.7}$$

The next lemma constitutes the existence of the normal coordinates in the vicinity of points of the inlet $\Sigma_{\rm in}$.

Lemma 9.3. *Let vector fields $\mathbf{u}$ and $\mathbf{U}$ meet all requirements of Lemma 9.2 and $U_n = -\mathbf{U}(P) \cdot \mathbf{n} > N > 0$. Then there are $b > 0$ and $C > 0$, depending only on N, Ω and $M = \|\mathbf{u}\|_{C^1(\Omega)}$, with the following properties. There exists a mapping $y \to \mathbf{x}(y)$, which takes diffeomorphically the cube $Q_b = [-b, b]^3$ onto a neighborhood $\mathcal{O}_P$ of P and satisfies the equations*

$$\partial_{y_3}\mathbf{x}(y) = \mathbf{u}(\mathbf{x}(y)) \text{ in } Q_b, \quad \mathbf{x}(y_1, y_2, 0) \in \partial\Omega \cap \mathcal{O}_P \textit{ for } |y_2| \le a, \tag{9.8}$$

and the inequalities

$$\|\mathbf{x}\|_{C^1(Q_b)} + \|\mathbf{x}^{-1}\|_{C^1(\mathcal{O}_P)} \le C \quad |\mathbf{x}(y)| \le C|y|, \quad \mathbf{x}(0) = P. \tag{9.9}$$

The inclusions

$$B_P(\rho_i) \cap \Omega \subset \mathbf{x}(Q_b \cap \{y_3 > 0\}) \subset B_P(R_i) \cap \Omega, \tag{9.10}$$

hold true for $\rho_i = C^{-1}b$ and $R_i = Cb$.

Model equation. Let $\mathcal{P}_a$ be a standard parabolic neighborhood associated with the constant c_0 and satisfying all conditions of Definition 9.1. Consider the boundary value problem

$$\partial_{y_1}\varphi(y) + \sigma\varphi(y) = f(y) \text{ in } \mathcal{P}_a, \quad \varphi(y) = 0 \text{ for } y_1 = a^-(Y). \tag{9.11}$$

Lemma 9.4. *Let exponents r, s and the accretivity defect α meet all requirements Theorem 1.5, $\sigma > 1$, and $f \in H^{s,r}(\mathcal{P}_a) \cap L^\infty(\mathcal{P}_a)$. Then there is a constant c, depending only on a, c_0, r, s such that a solution to problem* (9.11) *admits the estimates*

$$\|\varphi\|_{H^{s,r}(\mathcal{P}_a)} \leq c\big(\sigma^{-1}\|f\|_{H^{s,r}(\mathcal{P}_a)} + \wp(s, r\sigma)\|f\|_{L^\infty(\mathcal{P}_a)}\big), \quad \|\varphi\|_{L^\infty(\mathcal{P}_a)} \leq \sigma^{-1}\|f\|_{L^\infty(\mathcal{P}_a)} \tag{9.12}$$

where

$$\wp(s, r, \sigma) = (1+\sigma)^{-1+\alpha} \text{ for } rs \neq 1, 2, \quad \wp(s, r, \sigma) = \sigma^{-1+\alpha}(1+\log\sigma)^{1/r} \text{ for } rs = 1, 2,$$

the accretivity defect $\alpha = \max\{0, s - 1/r, 2s - 3/r\}$.

The proof is given in Appendix B. Next, let us consider the following boundary value problem

$$\partial_{y_3}\varphi + \sigma\varphi = f \text{ in } Q_a^+ = [-a, a]^2 \times [0, a], \quad \varphi(y) = 0 \text{ for } y_3 = 0. \tag{9.13}$$

Lemma 9.5. *Let exponents r, s and the accretivity defect α meet all requirements of Theorem 1.5 and $\sigma > 1$. Then for any $f \in H^{s,r}(\mathcal{P}_a) \cap L^\infty(\mathcal{P}_a)$, problem* (9.13) *has a unique solution satisfying the inequality*

$$\|\varphi\|_{H^{s,r}(Q_a^+)} \leq c(r, s, a)\big(\sigma^{-1}\|f\|_{H^{s,r}(Q_a^+)} + \wp(s, r, \sigma)\|f\|_{L^\infty(Q_a^+)}\big). \tag{9.14}$$

Proof. The proof of Lemma 9.4 can be used also in this simple case. □

Local existence results. It follows from the conditions of Theorem 1.5 that the vector field $\mathbf{u}$ and the manifold Γ satisfy all assumptions of Lemma 9.2. Therefore, there exist positive numbers a, ρ and R, depending only on Ω and $\|\mathbf{u}\|_{C^1(\Omega)}$, such that for all $P \in \Gamma$, the canonical diffeomorphism $\mathbf{x} : Q_a \mapsto \mathcal{O}_P$ is well-defined and meet all requirements of Lemma 9.2. Fix an arbitrary point $P \in \Gamma$ and consider the boundary value problem

$$\mathbf{u} \cdot \nabla\varphi + \sigma\varphi = f \text{ in } \mathcal{O}_P, \quad \varphi = 0 \text{ on } \Sigma_{\text{in}} \cap \mathcal{O}_P. \tag{9.15}$$

Lemma 9.6. *Assume that r, s, α and $\mathbf{U}$ satisfy all conditions of Theorem 1.5 and $\|\mathbf{u}\|_{C^1(\Omega)} \leq M$. Then there exists $\sigma^* > 1$, depending on Ω, s, r, and M such that for any $f \in C^1(\Omega)$ and $\sigma > \sigma^*$, problem* (9.15) *has a solution satisfying the inequalities*

$$|\varphi|_{s,r,B_P(\rho)} \leq c(\sigma^{-1}|f|_{s,r,B_P(R_c)} + \wp(s, r, \sigma)\|f\|_{L^\infty(B_P(R_c))}), \quad \|\varphi\|_{L^\infty(B_P(\rho))} \leq \sigma^{-1}\|f\|_{L^\infty(B_P(R))}, \tag{9.16}$$

where the constant c depends only on $\partial\Omega$, $\mathbf{U}$, M, s, r, and ρ, R are determined by Lemma 9.2. Moreover the solution is uniquely defined in the ball $B_P(\rho)$ and coincides with the weak solution to problem (9.15) *defined by Proposition 1.2*

Proof. We transform equation (9.15) using the normal coordinates y given by Lemma 9.2. Set $\overline{\varphi}(y) = \varphi(\mathbf{x}(y))$ and $\overline{f}(y) = f(\mathbf{x}(y))$. Next note that equation (9.3) implies the identity $\mathbf{u}\nabla_x\varphi = \partial_{y_1}\overline{\varphi}(y)$. Therefore the function $\overline{\varphi}(y)$ satisfies the following equation and boundary conditions

$$\partial_{y_1}\overline{\varphi} + \sigma\overline{\varphi} = \overline{f} \text{ in } \mathcal{P}_a, \quad \overline{\varphi} = 0 \text{ on } \Sigma_{in}^y, \tag{9.17}$$

where Σ_{in}^y is the set of all points $y = (y_1, Y) \in \partial\mathcal{P}_a$ such that $y_1 = a^-(Y) > -a$ is given by Definition 9.1. Next consider the boundary value problem

$$\partial_{y_1}\tilde{\varphi} + \sigma\tilde{\varphi} = \overline{f} \text{ in } \mathcal{P}_a, \quad \tilde{\varphi}(y) = 0 \text{ for } y_1 = a^-(Y). \tag{9.18}$$

It follows from Lemma 9.4, that $\tilde{\varphi}$ satisfies the inequality

$$\|\tilde{\varphi}\|_{H^{s,r}(\mathcal{P}_a)} \le c(\sigma^{-1}\|\overline{f}\|_{H^{s,r}(\mathcal{P}_a)} + \wp(s,r,\sigma)\|\overline{f}\|_{L^\infty(\mathcal{P}_a)}) \le c(\sigma^{-1}\|f\|_{H^{s,r}(\Omega)} + \wp(s,r,\sigma)\|f\|_{L^\infty(\Omega)})$$

Here the constant c depends only on M, r, s, $\mathbf{U}$ and $\partial\Omega$. It follows from Definition 9.1 that $\Sigma_{in}^y \subset \{y_1 = a^-(Y)\}$ and solutions to problems (9.17) and (9.18) coincide in the domain G_a determined by Definition 9.1.

If $\varphi^* \in L^\infty(\Omega)$ is a weak solution to problem (1.7), then the function $\hat{\varphi}(y) = \varphi^*(\mathbf{x}(y))$ satisfies the equation

$$\partial_{y_1}\hat{\varphi} + \sigma\hat{\varphi} = \overline{f} \text{ in } \mathcal{D}'(\mathcal{P}_a),$$

which is understood in the sense of distribution. It follows from this that the functions $\hat{\varphi}, \partial_{y_1}\hat{\varphi} \in L^\infty(\mathcal{P}_a)$ are continuous with respect to y_1. In particular, the trace $\hat{\varphi}(a^-(Y), Y) = \lim\limits_{y_1 \searrow a^-(Y)} \hat{\varphi}(y_1, Y)$ is well defined. On the other hand, by Proposition (1.2), the function φ^* is continuous and vanishes at Σ_{in}. Since $\mathbf{x}(\Sigma_{in}^y) = \Sigma_{\text{in}} \cap \mathcal{O}_P$, we conclude from this that the function $\hat{\varphi}$ vanishes at Σ_{in}^y and coincides with $\overline{\varphi}$ in the domain G_a. Hence $\varphi^* = \varphi$ in $B_P(\rho) \cap \Omega$, and the lemma follows. □

In order to formulate the similar result for interior points of inlet we introduce the set

$$\Sigma'_{\text{in}} = \{x \in \Sigma_{\text{in}} : \quad \text{dist } (x, \Gamma) \ge \rho/3), \tag{9.19}$$

where the constant ρ is given by Lemma 9.2. It is clear that

$$\inf_{P \in \Sigma'_{\text{in}}} \mathbf{U}(P) \cdot \mathbf{n}(P) \ge N > 0,$$

where the constant N depends only on M, $\mathbf{U}$, and $\partial\Omega$. It follows from Lemma 9.3 that there are positive numbers b, ρ_i, and R_i such that for for each $P \in \Sigma'_{\text{in}}$, the canonical diffeomorphism $\mathbf{x} : Q_b \mapsto \mathcal{O}_P \subset B_P(R_i)$ is well-defined and satisfies the hypotheses of Lemma 9.3. The following lemma gives the local existence and uniqueness of solutions to the boundary value problem

$$\mathbf{u} \cdot \nabla\varphi + \sigma\varphi = f \text{ in } \mathcal{O}_P, \quad \varphi = 0 \text{ on } \Sigma_{\text{in}} \cap \mathcal{O}_P. \tag{9.20}$$

Lemma 9.7. *Suppose that the exponents s, r, α satisfy condition* (1.12). *Then for any $f \in C^1(\Omega)$, $\sigma > 1$ and $P \in \Sigma'_{\text{in}}$, problem* (9.15) *has a unique solution satisfying the inequalities*

$$\begin{aligned} |\varphi|_{s,r,B_P(\rho_i)} &\le c\wp(s,r,\sigma)\|f\|_{L^\infty(B_P(R_i))} + \sigma^{-1}\|f\|_{H^{s,r}(B_P(R_i))}), \\ \|\varphi\|_{C(B_P(\rho_i))} &\le \sigma^{-1}\|f\|_{L^\infty(B_P(R_i))}. \end{aligned} \tag{9.21}$$

where c depends on Σ, M, $\mathbf{U}$ and exponents s, r. Moreover, this solution coincides with a weak solution to problem (9.15) *given by Proposition 1.2*

Proof. Using the normal coordinates given by Lemma 9.3 we rewrite equation (9.20) in the form.

$$\partial_{y_3}\overline{\varphi} + \sigma\overline{\varphi} = \overline{f} \text{ in } Q_b, \quad \overline{\varphi} = 0 \text{ for } y_3 = 0.$$

Applying Lemma 9.4 and arguing as in the proof of Lemma 9.6 we obtain (9.21). □

Existence of solutions near inlet. We are now in a position to prove the local existence and uniqueness of solution for the boundary value problem (1.7) near the inlet. Let Ω_t be the t-neighborhood of the set $\Sigma_{\rm in}$,

$$\Omega_t = \{x \in \Omega : \ \text{dist } (x, \Sigma_{\rm in}) < t\}.$$

Lemma 9.8. *Let $t = \min\{\rho/2, \rho_i/2\}$ and $T = \max\{R, R_i\}$, where the constants ρ, ρ_i and R, R_i are defined by Lemmas 9.2 and 9.3, respectively. Then there exists a constant C, depending only on M, $\partial\Omega$, $\mathbf{U}$ and exponents s, r, such that for any $f \in C^1(\Omega)$, the boundary value problem*

$$\mathbf{u} \cdot \nabla\varphi + \sigma\varphi = f \text{ in } \Omega_t, \quad \varphi = 0 \text{ on } \Sigma_{\rm in} \tag{9.22}$$

has a unique solution φ_t satisfying the inequalities

(9.23)

$$\|\varphi_t\|_{H^{s,r}(\Omega_t)} \le C(\wp(s,r,\sigma)\|f\|_{L^\infty(\Omega_T)} + \sigma^{-1}\|f\|_{H^{s,r}(\Omega_T)}), \ \|\varphi_t\|_{L^\infty(\Omega_t)} \le \sigma^{-1}\|f\|_{L^\infty(\Omega_T)}.$$

Moreover, φ_t coincides with a weak solution to problem (9.15) *given by Proposition 1.2.*

Proof. There exists a covering of the characteristic manifold Γ by the finite collection of balls $B_{P_i}(\rho/4)$, $1 \le i \le m$, $P_i \in \Gamma$. Obviously, the balls $B_{P_i}(\rho)$ cover the set

$$\mathcal{V}_\Gamma = \{x \in \Omega : \ \text{dist } (x, \Gamma) < \rho_c/2\}.$$

By virtue of Lemma 9.6 for any $P \in \Gamma$, a solution to problem (9.23) is uniquely determined in some neighborhood of P containing the ball $B_P(\rho)$. Hence it suffices to prove estimates (9.23). To this end note that by equality (1.2) we have

$$|\varphi|^r_{s,r,\mathcal{V}_\Gamma} = \int\limits_{(V_\Gamma)^2} |x-y|^{-3-rs}|\varphi(x)-\varphi(y)|^r dxdy \le$$

$$\int\limits_{(\mathcal{V}_\Gamma)^2\cap\{|x-y|<\rho/2\}} |x-y|^{-3-rs}|\varphi(x)-\varphi(y)|^r dxdy + c\rho^{-3-rs}\|\varphi\|^r_{L^\infty(\mathcal{V}_\Gamma)} \ \text{meas}(\mathcal{V}_\Gamma)^2.$$

Since any pair of points $x, y \in \mathcal{V}_\Gamma$ with $|x-y| < \rho/2$ belongs to some ball $B_{P_i}(\rho)$, the first term in the right- hand side of this inequality does not exceed the sum

$$\sum_i \int\limits_{B_{P_i}(\rho)^2} |x-y|^{-3-rs}|f(x)-f(y)|^r dxdy = \sum_i |\varphi|^r_{s,r,B_{P_i}(\rho)}$$

which leads to the estimate

$$\|\varphi\|_{H^{s,r}(\mathcal{V}_\Gamma)} = |\varphi|_{s,r,\mathcal{V}_\Gamma} + \|\varphi\|_{L^r(\mathcal{V}_\Gamma)} \le c\sum_i \|\varphi\|_{H^{s,r}(B_{P_i}(\rho))} + c\|\varphi\|_{L^\infty(\mathcal{V}_\Gamma)}, \tag{9.24}$$

where c depends on s, r and ρ, i.e., on s, r, $\mathbf{U}$, $\partial\Omega$ and M. By Lemma 9.6 in each of such balls, a solution to problem (9.22) satisfies inequalities (9.16), which leads to the estimate

(9.25)

$$\|\varphi\|_{H^{s,r}(\mathcal{V}_\Gamma)} \le c\wp(s, r\sigma)\sum_i \|f\|_{C(B_{P_i}(R))} + c\sigma^{-1}\sum_i \|f\|_{H^{s,r}(B_{P_i}(R))} + c\|\varphi\|_{L^\infty(\mathcal{V}_\Gamma)}.$$

On the other hand, we have $\|\varphi\|_{L^\infty(B_{P_i}(\rho))} \leq \sigma^{-1}\|f\|_{C(B_{P_i}(R))}$. Moreover, since $B_{P_i}(\rho)\subset\Omega_T$ we have

$$\wp(s,r\sigma)\sum_i \|f\|^r_{C(B_{P_i}(R))}+\sigma^{-1}\sum_i \|f\|_{H^{s,r}(B_{P_i}(R))}\leq \wp(s,r\sigma)\|f\|_{C(\Omega_T)}+m\sigma^{-1}\|f\|_{H^{s,r}(\Omega_T)}$$

¿From this and (9.25) we finally obtain the estimates for solution to problem (9.22) in the neighborhood of the characteristic manifold Γ,

$$\|\varphi\|_{H^{s,r}(\mathcal{V}_\Gamma)}\leq c\wp(s,r\sigma)\|f\|_{C(\Omega_T)}+c\sigma^{-1}\|f\|_{H^{s,r}(\Omega_T)}, \tag{9.26}$$

where c depends only on M, $\partial\Omega$, $\mathbf{U}$ and s,r.

Our next task is to obtain the similar estimate in the neighborhood of the compact $\Sigma'_{\text{in}}\subset\Sigma_{\text{in}}$. To this end, we introduce the set

$$\mathcal{V}_{\text{in}}=\{x\in\Omega:\ \text{dist }(x,\Sigma'_{\text{in}})<\rho_i/2\},$$

where Σ'_{in} is given by (9.19). Let $B_{P_k}(\rho_i/4)$, $1\leq k\leq m$, be a minimal collection of balls of radius $\rho_i/4$ covering Σ'_{in}. It is clear that the balls $B_{P_k}(\rho_i)$ cover the set $\mathcal{V}_{\text{in}}$. Arguing as in the proof of (9.24) we obtain

$$\|\varphi\|_{H^{s,r}(\mathcal{V}_{\text{in}})}\leq\sum_k|\varphi|_{s,r,B_{P_k}(\rho_i)}+c\|\varphi\|_{L^\infty(\mathcal{V}_{\text{in}})}$$

¿From this and Lemma 9.7 we obtain

$$\|\varphi\|_{H^{s,r}(\mathcal{V}_{\text{in}})}\leq c\wp(s,r\sigma)\sum_k\|f\|^r_{C(B_{P_k}(R_i))}+c\sigma^{-1}\sum_k|f|^r_{s,r,B_{P_k}(R_i)}.$$

Thus, we get

$$|\varphi|_{s,r,\mathcal{V}_{in}}\leq c\wp(r,s,\sigma)\|f\|_{C(\Omega_T)}+c\sigma^{-1}|f|_{s,r,\Omega_T}.$$

Since $\mathcal{V}_\Gamma$ and $\mathcal{V}_{\text{in}}$ cover Ω_t, this inequality along with inequalities (9.26) yields (9.23), and the lemma follows. □

Partition of unity. Let us turn to the analysis of general problem

$$\mathscr{L}\varphi:=\mathbf{u}\cdot\nabla\varphi+\sigma\varphi=f\text{ in }\Omega,\quad\varphi=0\text{ on }\Sigma_{\text{in}}. \tag{9.27}$$

We split the weak solution $\varphi\in L^\infty(\Omega)$ to problem (9.27) into two parts, namely the local solution φ_t, determined by Lemma 9.8, and the remainder vanishing near inlet. To this end fix a function $\eta\in C^\infty(\mathbb{R})$ such that

$$0\leq\eta'\leq 3,\quad\eta(u)=0\text{ for }u\leq 1\text{ and }\eta(u)=1\text{ for }u\geq 3/2, \tag{9.28}$$

and introduce the one-parametric family of smooth functions

$$\chi_t(x)=\frac{1}{t^3}\int_{\mathbb{R}^3}\varpi\Big(\frac{2(x-y}{t}\Big)\eta\Big(\frac{\text{dist }(y,\Sigma_{\text{in}})}{t}\Big)\,dy. \tag{9.29}$$

where $\varpi\in C^\infty(\mathbb{R}^3)$ is a standard mollifying kernel. It follows that χ_t is a C^∞ function with $|\nabla\chi_t|\leq c/t$ and

$$\chi_t(x)=0\text{ for dist }(x,\Sigma_{\text{in}})\leq t/2,\quad\chi_t(x)=1\text{ for dist }(x,\Sigma_{\text{in}})\geq 2t. \tag{9.30}$$

Now fix a number t satisfying all assumptions of Lemma 9.8 and set

$$\varphi(x)=(1-\chi_{t/2}(x))\varphi_t(x)+\phi(x). \tag{9.31}$$

By virtue of (9.30) and Lemma 9.8, the function $\phi\in L^\infty(\Omega)$ vanishes in $\Omega_{t/4}$ and satisfies in a weak sense to the equations

$$\mathbf{u}\cdot\nabla\phi+\sigma\phi=\chi_{t/2}f+\varphi_t\mathbf{u}\cdot\nabla\chi_{t/2}=:F\text{ in }\Omega,\ \ \phi=0\text{ on }\Sigma_{\text{in}}.$$

Next introduce new vector field $\tilde{\mathbf{u}}(x) = \chi_{t/8}(x)\mathbf{u}(x)$. It easy to see that $\chi_{t/8} = 1$ on the support of ϕ and hence the function ϕ is also a weak solution to the modified transport equation

$$\tilde{\mathscr{L}}\phi := \tilde{\mathbf{u}} \cdot \nabla\phi + \sigma\phi = F \text{ in } \Omega. \tag{9.32}$$

The advantage gained here is that the topology of integral lines of the modified vector field $\tilde{\mathbf{u}}$ drastically differs from the topology of integral lines of $\mathbf{u}$. The corresponding outgoing and characteristic sets have the other structure and $\tilde{\Sigma}_{\text{in}} = \emptyset$. In particular, equation (9.32) does not require boundary conditions. Finally note that C^1-norm of the modified vector fields has the majorant

$$\|\tilde{\mathbf{u}}\|_{C^1(\Omega)} \le M(1 + c_1 t^{-1}), \tag{9.33}$$

The following lemma constitutes the existence and uniqueness of solutions to the modified equation.

Lemma 9.9. *Suppose that*

$$\sigma > \sigma^*, \quad \sigma^* = 16M(1 + c_1 t^{-1}) + 16, \quad M = \|\mathbf{u}\|_{C^1(\Omega)}, \tag{9.34}$$

and $0 \le s \le 1$, $r > 1$. *Then for any* $F \in H^{s,r}(\Omega) \cap L^\infty(\Omega)$, *equation* (9.32) *has a unique weak solution* $\phi \in H^{s,r}(\Omega) \cap L^\infty(\Omega)$ *such that*

$$\|\phi\|_{H^{s,r}(\Omega)} \le c\sigma^{-1}\|F\|_{H^{s,r}(\Omega)}, \quad \|\phi\|_{L^\infty(\Omega)} \le \sigma^{-1}\|F\|_{L^\infty(\Omega)}, \tag{9.35}$$

where c *depends only on* r.

Proof. Without any loss of generality we can assume that $F \in C^1(\Omega)$. By virtue of (9.33) and (9.34), the vector field $\tilde{\mathbf{u}}$ and σ meet all requirements of Proposition 1.3. Hence equation (9.32) has a unique solution $\phi \in H^{1,\infty}(\Omega)$. For $i = 1, 2, 3$ and $\tau > 0$, define the finite difference operator

$$\delta_{i\tau}\phi = \frac{1}{\tau}\big(\phi(x + \tau\mathbf{e}_i) - \phi(x)\big).$$

It is easily to see that

$$\tilde{\mathbf{u}} \cdot \nabla\delta_{i\tau}\phi + \sigma\delta_{i\tau}\phi = \delta_{i\tau}F - \delta_{i\tau}\tilde{\mathbf{u}} \cdot \nabla\phi(x + \tau\mathbf{e}_i) \text{ in } \Omega \cap (\Omega - \tau\mathbf{e}_i). \tag{9.36}$$

Next set $\eta_h(x) = \eta\big(\text{dist }(x, \partial\Omega)/h\big)$, where η is defined by (9.28). Since $\tilde{\Sigma}_{\text{in}} = \emptyset$, the inequality

$$\limsup_{h\to 0} \int\limits_\Omega g\tilde{\mathbf{u}} \cdot \nabla\eta_h(x)\, dx \le 0 \tag{9.37}$$

holds true for all nonnegative functions $g \in L^\infty(\Omega)$. Choosing $h > \tau$, multiplying both sides of equation (9.36) by $\eta_h|\delta_{i\tau}\phi|^{r-2}\delta_{i\tau}\phi$ and integrating the result over $\Omega \cap (\Omega - \tau\mathbf{e}_i)$ we obtain

$$\int\limits_{\Omega\cap(\Omega-\tau\mathbf{e}_i)} \eta_h|\delta_{i\tau}\phi|^r(\sigma - \frac{1}{r}\operatorname{div}\tilde{\mathbf{u}})\, dx - \int\limits_{\Omega\cap(\Omega-\tau\mathbf{e}_i)} |\delta_{i\tau}\phi|^r\tilde{\mathbf{u}} \cdot \nabla\eta_h\, dx =$$
$$\int\limits_{\Omega\cap(\Omega-\tau\mathbf{e}_i)} \big(\delta_{i\tau}F - \delta_{i\tau}\tilde{\mathbf{u}} \cdot \nabla\phi(x + \tau\mathbf{e}_i)\big)\eta_h|\delta_{i\tau}\phi|^{r-2}\delta_{i\tau}\phi\, dx.$$

Letting $\tau \to 0$ and then $h \to 0$ and using inequality (9.37) we obtain

$$\int_\Omega |\partial_{x_i}\phi|^r(\sigma - \frac{1}{r}\operatorname{div}\tilde{\mathbf{u}})\,dx \le \int_\Omega \big(\partial_{x_i}F - \partial_{x_i}\tilde{\mathbf{u}}\cdot\nabla\phi\big)|\partial_{x_i}\phi|^{r-2}\partial_{x_i}\phi\,dx. \tag{9.38}$$

Next note that

$$\sum_i \partial_{x_i}\tilde{\mathbf{u}}\cdot\nabla\phi\big|\partial_{x_i}\phi|^{r-2}\partial_{x_i}\phi \le 3\|\tilde{\mathbf{u}}\|_{C^1(\Omega)}\sum_i |\partial_{x_i}\phi|^r.$$

On the other hand, since $1/r + 3 \le 4$, inequalities (9.33) and (9.34) imply

$$\sigma - (\frac{1}{r}+3)\|\tilde{\mathbf{u}}\|_{C^1(\Omega)} \ge \sigma - \sigma^* \ge 1.$$

¿From this we conclude that

$$(\sigma-\sigma^*)\sum_i\int_\Omega |\partial_{x_i}\phi|^r\,dx \le \sum_i\int_\Omega |\partial_{x_i}\phi|^{r-1}|\partial_{x_i}F|\,dx \le c\|\nabla\phi\|^{r-1}_{L^r(\Omega)}\|\nabla F\|_{L^{r'}(\Omega)}$$

which leads to the estimate

$$\|\nabla\phi\|_{L^r(\Omega)} \le c(r)\sigma^{-1}\|\nabla F\|_{L^r(\Omega)} \text{ for } \sigma > \sigma^*(M,r). \tag{9.39}$$

Next multiplying both sides of (9.32) by $|\phi|^{r-2}\eta_h$ and integrating the result over Ω we get the identity

$$\int_\Omega(\sigma - r^{-1}\operatorname{div}\tilde{\mathbf{u}})\eta_h|\phi|^r\,dx - \int_\Omega |\phi|^r\tilde{\mathbf{u}}\cdot\nabla\eta_h\,dx = \int_\Omega F\eta_h|\phi|^{r-2}\phi\,dx.$$

The passage $h \to 0$ gives the inequality

$$\int_\Omega(\sigma - r^{-1}\operatorname{div}\tilde{\mathbf{u}})|\phi|^r\,dx \le \int_\Omega |F||\phi|^{r-1}\,dx.$$

Recalling that $\sigma - r^{-1}\operatorname{div}\tilde{\mathbf{u}} \ge \sigma-\sigma^*$ we finally obtain

$$\|\phi\|_{L^r(\Omega)} \le c(r)\sigma^{-1}\|F\|_{L^r(\Omega)}. \tag{9.40}$$

Inequalities (9.39) and (9.40) imply estimate (9.35) for $s = 0,1$. Hence for $\sigma > \sigma^*$, the linear operator $\tilde{\mathscr{L}}^{-1} : F \mapsto \phi$ is continuous in the Banach spaces $H^{0,r}(\Omega)$ and $H^{1,r}(\Omega)$ and its norm does not exceed $c(r)\sigma^{-1}$. Recall that $H^{s,r}(\Omega)$ is the interpolation space $[L^r(\Omega), H^{1,r}(\Omega)]_{s,r}$. From this and the interpolation theory we conclude that inequality (9.35) is fulfilled for all $s \in [0,1]$, which completes the proof. □

We are now in a position to complete the proof of Theorem 1.5. Fix $\sigma > \sigma^*$, where the constant σ^* depends only on Σ, $\mathbf{U}$ and $\|\mathbf{u}\|_{C^1(\Omega)}$, and it is defined by (9.34). Without any loss of generality we can assume that $f \in C^1(\Omega)$. The existence and uniqueness of a weak bounded solution for $\sigma > \sigma^*$, follows from Lemma 1.2. Therefore, it suffices to prove estimate (1.13) for $\|\varphi\|_{H^{s,r}(\Omega)}$. Since $H^{s,r}(\Omega)\cap L^\infty(\Omega)$ is the Banach algebra, representation (9.31) together with inequality (9.30) implies

$$\|\varphi\|_{H^{s,r}(\Omega)} \le c(\|\varphi_t\|_{H^{s,r}(\Omega_t)} + \|\varphi_t\|_{L^\infty(\Omega_t)}) + c\|\phi\|_{H^{s,r}(\Omega)}. \tag{9.41}$$

On the other hand, Lemma 9.9 along with (9.32) yields

$$\begin{aligned}\|\phi\|_{H^{s,r}(\Omega)} \le c\sigma^{-1}\|F\|_{H^{s,r}(\Omega)} &\le c\sigma^{-1}\|\chi_{t/2}f\|_{H^{s,r}(\Omega)} + \sigma^{-1}\|\varphi_t\mathbf{u}\cdot\nabla\chi_{t/2}\|_{H^{s,r}(\Omega)}\\ &\le c\sigma^{-1}(\|f\|_{H^{s,r}(\Omega)} + \|\varphi_t\|_{H^{s,r}(\Omega_t)}).\end{aligned}$$

Substituting this estimates into (9.41) we arrive at the inequality

$$\|\varphi\|_{H^{s,r}(\Omega)} \le c\big(\sigma^{-1}\|\varphi_t\|_{H^{s,r}(\Omega)} + \|\varphi_t\|_{L^\infty(\Omega_t)} + \sigma^{-1}\|f\|_{H^{s,r}(\Omega_t)} + \sigma^{-1}\|f\|_{L^\infty(\Omega)}\big),$$

which along with (9.23) leads to the estimate (1.13), and the theorem follows.

9.2. **Proof of Theorem 1.7.** It follows from conditions of Theorem 1.7 that a solution φ meets al requirements of Theorem 1.5 and admits estimate (1.13). By Conditions of Theorem 1.7, we have $r(s/2) > 3$ and hence

$$\sigma^\kappa\|f\|_{L^\infty(\Omega)} \le \sigma^\kappa\|f\|_{H^{s/2,r}(\Omega)} \le \sigma^\kappa\|f\|^{1/2}_{H^{s,r}(\Omega)}\|f\|^{1/2}_{L^r(\Omega)} \le$$
$$\|f\|_{H^{s,r}(\Omega)} + c\sigma^{2\kappa}\|f\|_{L^r(\Omega)} \le c\|f\|_{s,r} + c\sigma^{2\kappa}\|f\|_{L^r(\Omega)}.$$

Substituting this estimate in (1.13) we obtain (1.17) and the theorem follows.

10. Proof of Theorem 1.9

10.1. **Representation of harmonic vector fields.** First we prove the following auxiliary result on the representation of harmonic vector fields in the unit ball.

Theorem 10.1. *Let $v \in C^\infty(B)$ be a harmonic function in the unit ball $B = \{|y| < 1\}$. Assume that a vector field $\mathbf{W} \in C^\infty(B)$ satisfies the following equations and boundary conditions*

$$\Delta\mathbf{W} = 0 \text{ in } \Omega, \quad \mathbf{W} = \partial_n\psi\,\mathbf{n} \text{ on } \partial\Omega, \tag{10.1}$$

$$\Delta\psi = v \text{ in } \Omega, \quad \psi = 0 \text{ on } \partial\Omega. \tag{10.2}$$

Then the vector field $\mathbf{W}$ has the representation

$$\operatorname{div}\mathbf{W}(y) = \frac{1}{2}v(y) + \mathscr{V}v(y) \text{ in } B, \text{ where } \mathscr{V}v(y) = \frac{1}{4}\int_0^1 t^{-1/2}v(ty)\,dt, \tag{10.3}$$

and for any $r \in (0,1)$ and integer $s \ge 0$,

$$\|\mathscr{V}v\|_{H^{s+1,r}(B)} \le c(r,s)\|v\|_{H^{s,r}(B)}. \tag{10.4}$$

Proof. We split the proof into the sequence of lemmas.

Lemma 10.2. *Under the assumptions of Theorem 10.1,*

$$\partial_n\psi \equiv \nabla\psi\cdot\mathbf{n} = \frac{1}{2}\int_0^1 t^{1/2}v(ty)\,dt \text{ for } y \in \partial B. \tag{10.5}$$

Proof. Let us consider the classical identity

$$\psi(y) = \int_B G(y,x)v(x)\,dx, \text{where } G(y,x) = \frac{1}{4\pi}\Big(\frac{|x|}{||x|^2y - x|} - \frac{1}{|x-y|}\Big).$$

Direct calculations show that the normal derivative of the Green function coincides with the Poisson kernel

$$\partial_{n_y}G(y,x) \equiv \nabla_y G(y,x)\cdot y = \frac{1-|x|^2}{4\pi|x-y|^3} \text{ for } |y| = 1,$$

Using the denotations

$$x = \rho\xi, \quad y = \rho\varsigma, \quad \xi,\varsigma \in \partial B, \quad cos\,\theta = \varsigma\cdot\xi,$$

in which ξ and ς are arbitrary points of ∂B, we can rewrite the expression of the normal derivative of ψ in the form

$$\partial_n\psi(\varsigma) = \int_0^1 \rho^2\, d\rho \Big\{ \frac{1}{4\pi} \int_{\partial B} \frac{1-\rho^2}{|1+\rho^2-2\rho\cos\,\theta)|^{3/2}} v(\rho\xi)\, d\xi \Big\} \text{ for all } \varsigma \in \partial B.$$

Recall that for any harmonic function u in the unit ball,

$$u(\rho\varsigma) = \frac{1}{4\pi} \int_{\partial B} \frac{1-\rho^2}{|1+\rho^2-2\rho\cos\,\theta)|^{3/2}} u(\xi)\, d\xi \text{ for all } \varsigma \in \partial B,$$

which leads to the identity

$$\frac{1}{4\pi} \int_{\partial B} \frac{1-\rho^2}{|1+\rho^2-2\rho\cos\,\theta)|^{3/2}} v(\rho\xi)\, d\xi = v(\rho^2\varsigma).$$

Thus, we get

$$\partial_n\psi(y) = \int_0^1 \rho^2 v(\rho^2 y)\, d\rho = \frac{1}{2}\int_0^1 t^{1/2} v(ty)\, dt \text{ for all } y \in \partial B$$

and the lemma follows. □

Lemma 10.3. *Under the assumptions of Theorem 10.1, the vector field* $\mathbf{W}$ *satisfies equality* (10.3).

Proof. By virtue of Lemma 10.2, we have

$$\mathbf{W}(y) = \frac{1}{2}\int_0^1 t^{1/2}\mathbf{W}_t(y)\, dt \text{ for } y \in B, \tag{10.6}$$

where the vector fields $\mathbf{W}_t$ satisfy the equations and boundary conditions

$$\Delta\mathbf{W}_t = 0 \text{ in } B, \quad \mathbf{W}_t = v_t(y)y \text{ on } \partial B, \text{ where } v_t(y) = v(ty).$$

Since $\Delta(v_t y) = 2\nabla v_t$, we have $\mathbf{W}_t = v_t y - 2\Phi_t$, where the vector field Φ_t is a solution to the boundary value problem

$$\Delta\Phi_t = \nabla v_t \text{ in } B, \quad \Phi_t = 0 \text{ on } \partial B.$$

Thus, we get

$$\operatorname{div}\mathbf{W}_t(y) = \operatorname{div}\big(v_t(y)y\big) - 2\operatorname{div}\Phi_t(y). \tag{10.7}$$

On the other hand, since the components $\Phi_{t,i}$ of the vector field Φ_t vanish at ∂B, we have $\partial_{y_i}\Phi_{t,i} = \partial_n\Phi_{t,i}n_i$ on ∂B, which leads to the boundary condition for $\operatorname{div}\Phi_t$

$$\operatorname{div}\Phi_t = \partial_n\Phi_t \cdot \mathbf{n} \text{ on } \partial B. \tag{10.8}$$

Since $\partial_{y_i}v_t$ is a harmonic function in B, identity (10.5), with ψ and v replaced by Φ_{ti} and $\partial_{y_i}v_t$, respectively, yields the representation

$$\partial_n\Phi_{ti}(y) = \frac{1}{2}\int_0^1 \tau^{1/2}[\partial_{y_i}v_t](\tau y)\, d\tau \text{ for } y \in \partial B,$$

which along with the equality (10.8) leads to the identity

$$\operatorname{div}\Phi_t = \frac{1}{2}\int_0^1 \tau^{1/2}[\nabla v_t](\tau y)\cdot y\, d\tau \text{ for } y \in \partial B\ .$$

Recalling that $\operatorname{div}\Phi_t$ and $\nabla v_t(y)\cdot y$ are harmonic in the unit ball we obtain

$$\operatorname{div}\Phi_t = \frac{1}{2}\int_0^1 \tau^{1/2}[\nabla v_t](\tau y)\cdot y\, d\tau \text{ for } y \in B\ .$$

Noting that $[\nabla v_t](\tau y)\cdot y = \partial_\tau v_t(\tau y)$, we conclude from this that

$$\operatorname{div}\Phi_t(y) = \frac{1}{2}v_t(y) - \frac{1}{4}\int_0^1 \tau^{-1/2} v_t(\tau y)\, d\tau \text{ in } B.$$

Substituting this equality into (10.7) we arrive at the identity

$$\operatorname{div}\mathbf{W}_t(y) = 2v_t(y) + \nabla v_t(y)\cdot y + \frac{1}{2}\int_0^1 \tau^{-1/2} v_t(\tau y)\, d\tau \text{ in } B,$$

which along with (10.6) implies
(10.9)

$$\operatorname{div}\mathbf{W}(y) = \int_0^1 t^{1/2} v_t(y)\, dt + \frac{1}{2}\int_0^1 t^{1/2}[\nabla v_t](y)\cdot y\, dt + \frac{1}{4}\int_0^1\int_0^1 t^{1/2}\tau^{-1/2} v_t(\tau y)\, dt d\tau.$$

Using the identities $\nabla v_t\cdot y = t\partial_t[v(ty)]$ and $v_t(\tau y) = v(t\tau y)$ we can rewrite the integrals in the right-hand side in the form

$$\int_0^1 t^{1/2}\nabla v_t(y)\cdot y\, dt = v(y) - \frac{3}{2}\int_0^1 t^{1/2} v(ty)\, dt, \quad \int_0^1\int_0^1 t^{1/2}\tau^{-1/2} v(t\tau y)\, dt d\tau = \int_0^1 (z^{-1/2} - z^{1/2}) v(zy)\, dz.$$

Substituting these equalities into (10.9) we arrive at (10.3), and the lemma follows.
□

Our next task is to prove estimate (10.4) for the remainder in representation (10.3). We obtain this estimate, as a consequence of the following result on weighted estimates for the harmonic functions in the unit ball.

Lemma 10.4. *Assume that an integrable function $a : [0,1] \mapsto \mathbb{R}$ satisfies the conditions*

$$|a(1)| \le M, \quad \int_0^1 (1-t)^{-1}|a(t) - a(1)|\, dt \le M, \text{ and } \mathscr{F}_a v(y) = \int_0^1 a(t) v(ty)\, dt.$$

Then for any $r \in (1,\infty)$ and integer $s \ge 0$, there exists a positive constant $c(r,s)$ such that the inequality

$$\|\mathscr{F}_a v\|_{H^{s+1,r}(B)} \le M c(r,s)\|v\|_{H^{s,r}(B)}$$

holds true for all harmonic functions $v \in C^\infty(B)$.

Proof. First we prove the lemma in the case $a(1)=0$. It easy to see that

$$\|v_t\|_{H^{s+1,r}(B)} \le t^{-3/r} c(r,s)\|v\|_{H^{s+1,r}(B_t)},$$

where $v_t = v(t\cdot)$, $B_t = tB$. Thus, we get

$$\|\mathscr{F}_a v\|_{H^{s+1,r}(B)} \le c(r,s)\int_0^1 |a(t)| t^{-3/r}\|v\|_{H^{s+1,r}(B_t)}\,dt. \tag{10.10}$$

Let us estimate $(s+1)$-norm of v in B_t by its s-norm in the ball B. Denote by $\eta\in C^\infty(\mathbb{R})$ the cut-off function, which vanishes on the interval $(-\infty,0]$ and is equal to 1 on the interval $[1,\infty)$. For any $t\in[0,1)$, set

$$\eta_t(y) = \eta\big((1-t^2)^{-1}(1-|y|^2)\big), \quad y\in B.$$

It is clear that η_t vanishes on ∂B and is equal to 1 in the ball B_t. Next note that for any test function $\varphi\in C_0^\infty(B)$,

$$\Big|\int_B (v\Delta\eta_t + 2\nabla v\nabla\eta_t)\varphi\,dt\Big| = \Big|\int_B v(\varphi\Delta\eta_t - 2\operatorname{div}(\varphi\nabla\eta_t)v)\,dt\Big| \le$$
$$\frac{c}{(1-t^2)^2}\int_{B\setminus B_t}|v||\varphi|\,dy + \frac{c}{(1-t^2)}\int_{B\setminus B_t}|v||\nabla\varphi|\,dy \le$$
$$\frac{c}{(1-t)}\Big(\int_B |v|\frac{|\varphi|}{\text{dist }(y,\partial B)}\,dy + \int_B |v||\nabla\varphi|\,dy\Big) \le$$
$$c\|v\|_{L^r(B)}(\|\varphi/\text{dist }(y,\partial B)\|_{L^{r'}(B)} + \|\nabla\varphi\|_{L^{r'}(B)}) \le c\|v\|_{L^r(B)}\|\varphi\|_{H^{1,r'}(B)},$$

which gives the estimate

$$\|v\Delta\eta_t + 2\nabla v\nabla\eta_t\|_{H^{-1,r}(B)} \le c(r)(1-t)^{-1}\|v\|_{L^r(B)}. \tag{10.11}$$

Since the function $\eta_t v$ satisfies the following equation along with the boundary conditions

$$\Delta(\eta_t v) = v\Delta\eta_t + 2\nabla v\nabla\eta_t \text{ in } B, \quad \eta_t v = 0 \text{ on } \partial B,$$

we obtain

$$\|v\|_{H^{1,r}(B_t)} \le c(r)\|\eta_t v\|_{H^{1,r}(B)} \le c(r)(1-t)^{-1}\|v\|_{L^r(B)}.$$

Applying these arguments to the harmonic functions $\partial^\alpha v$, $|\alpha|\le s$, we conclude that for any integer $s\ge 0$,

$$\|v\|_{H^{1+s,r}(B_t)} \le c(r)(1-t)^{-1}\|v\|_{H^{s,r}(B)}. \tag{10.12}$$

Moreover, if we consider the chain of the balls $B_{1/2}\subset B_{5/8}\subset B_{6/8}\subset B_{7/8}\subset B$, then we get the estimate

$$\|v\|_{C^{s+1}(B_{1/2})} \le c(s,r)\|v\|_{H^{s+4,r}(B_{1/2})} \le 8^4 c(s,r)\|v\|_{H^{s,r}(B)}. \tag{10.13}$$

Combining (10.10) with (10.12), (10.13) and using the classical inequality $\|v\|_{H^{s,r}(B_t)} \leq ct^{3/r}\|v\|_{C^s(B_t)}$ we obtain

$$\|\mathscr{F}_a v\|_{H^{s+1,r}(B)} \leq c\|v\|_{C^{s+1}(B_{1/2})} \int_0^{1/2} |a(t)|\,dt + c\int_{1/2}^1 \|v\|_{H^{s+1,r}(B_t)} \leq$$

$$c\|v\|_{H^{s,r}(B)} \int_0^1 (1-t)^{-1}|a(t)|\,dt,$$

which gives the desired estimate in the case $a(1)=0$. Next note that by virtue of Lemma 10.2, the function $2u \equiv \mathscr{F}_{\sqrt{t}}v$ satisfies the following equations and the boundary conditions

$$\Delta u = 0, \quad \Delta\psi = v \text{ in } B,$$
$$u = \partial_n\psi \equiv \nabla\psi\cdot y, \quad \psi = 0 \text{ on } \partial B.$$

Since $\|\psi\|_{H^{s+2,r}(B)} \leq c(r,s)\|v\|_{H^{s,r}(B)}$ we have

$$\|\mathscr{F}_{\sqrt{t}}v\|_{H^{s+1,r}(B)} \leq c\|u\|_{H^{s+1,r}(B)} \leq c\|\nabla\psi\cdot y\|_{H^{s+1,r}(B)} \leq c\|v\|_{H^{s,r}(B)},$$

which gives the desired estimate in the case $a=\sqrt{t}$. The general case follows from the representation $a(t) = a(t) - a(1) + a(1)(1-\sqrt{t}) + a(1)\sqrt{t}$. $\square$

In order to complete the proof of Theorem 10.1 it remains to note that $\mathscr{V} = 4^{-1}\mathscr{F}_{1/\sqrt{t}}$ and the function $1/\sqrt{t}$ meets all requirements of Lemma 10.4. $\square$

The following theorem gives an extension of Theorem 10.1 in the case of an arbitrary bounded domain with the smooth boundary.

Theorem 10.5. *Let $\Omega \subset \mathbb{R}^3$ be a bounded domain with C^∞ boundary $\partial\Omega$, s be a nonnegative integer and $r \in (1,\infty)$ be given. Then for any harmonic function $u \in C^\infty(\Omega)$,*

$$\|\mathscr{A}u - u/2\|_{H^{s+1,r}(\Omega)} \leq c(r,s,\Omega)\|u\|_{H^{s,r}(\Omega)} \tag{10.14}$$

Proof. We reduce Theorem 10.5 to Theorem 10.1 using the change of independent variables and the partition of unity. To this end we note that

$$\mathscr{A}u = u - \operatorname{div}\mathbf{W}, \tag{10.15}$$

where a harmonic vector field $\mathbf{W}$ is given by a solution to the boundary value problem

$$\begin{gathered}\Delta\mathbf{W} = 0 \text{ in } \Omega, \quad \mathbf{W} = \partial_n\phi\,\mathbf{n} \text{ on } \partial\Omega,\\ \Delta\phi = u \text{ in } \Omega, \quad \phi = 0 \text{ on } \partial\Omega.\end{gathered} \tag{10.16}$$

We consider $\operatorname{div}\mathbf{W}$ as a linear operator on the linear space of smooth harmonic functions, and prove that the principal part of this operator is simply $u/2$. The proof of this fact naturally falls into three steps :

Step 1. For each diffeomorphism $y \to \mathbf{x}(y)$ of an open set subset of $\mathbb{R}^3$, we denote by $\mathbb{Y}$ and $\mathbf{g}$ the matrix-valued functions defined by the equalities

$$\mathbb{Y}(y) = \big(\mathbf{x}'(y)^{-1}\big)^*, \quad \mathbf{g}(y) = \mathbb{Y}^*\mathbb{Y}, \quad g = \det\mathbf{g}.$$

In particular, the elements of the fundamental matrix $\mathbf{g}$ are given by the formulae

$$g_{ij}(y) = [\nabla y_i](\mathbf{x}(y))\cdot[\nabla y_j](\mathbf{x}(y)).$$

For given $\rho > 0$, denote by B^+ the ball

$$B^+ = \{y \in \mathbb{R}^3 : |y - P^+| \le \rho, \quad P^= \{0,0,1)\}.$$

Lemma 10.6. *For an arbitrary point $P \in \partial\Omega$, there exist $\rho > 0$, a neighborhood U of P, and mapping $x \to \mathbf{y}(x)$ of class $C^\infty(U)$ such that $\mathbf{y}(x)$ takes diffeomorphically U onto the ball B^+, and*

$$\mathbf{y}(U \cap \Omega) = B \cap B^+, \quad \mathbf{y}(\partial\Omega \cap U) = \partial B \cap B^+.$$

Moreover corresponding fundamental matrix $\mathbf{g} \in C^\infty(B^+)$ has the representation

$$\mathbf{g}(y) = g^{1/3}\mathbf{I} + (1 - |y|^2)\mathbb{G}(y) \text{ in } B \cup B^+, \text{ where } \mathbb{G} \in C^\infty(B^+). \tag{10.17}$$

Proof To simplify our notation the diffeomorphisms $\mathbf{x}, \mathbf{y}$ are denoted by the same symbols as the variables x, y.

In some neighborhood U of P, the surface $\partial\Omega \cup U$ admits C^∞ conformal parameterization $x = x(q_1, q_2)$, $|q_i| < \rho$, such that $x(0,0) = P$ and

$$\partial_{q_i} x(q_1, q_2) \cdot \partial_{q_j} x(q_1, q_2) = Q(q_1, q_2)\delta_{ij},$$

where Q is a positive C^∞ function. In particular the outward normal $\mathbf{n}$ to $\partial\Omega \cap U$ is a smooth function of conformal coordinates q_i. For suitable choice of U and ρ, the mapping

$$q = (q_1, q_2, q_3) \to x(q_1, q_2) - q_3 Q(q_1, q_2)\mathbf{n}(q_1, q_2)$$

takes diffeomorphically the ball $B(\rho) = \{|q| < \rho\}$ onto U and

$$q(B(\rho)) \cap \{q_3 > 0\}) = U \cap \Omega, \quad q(B(\rho)) \cap \{q_3 = 0\}) = U \cap \partial\Omega.$$

The corresponding fundamental matrix $\mathbf{g}(q) = x'(q)^{-1}(x'(q)^{-1})^*$ coincides with $Q^{-1}(q)\mathbf{I}$ on the hyper-plane $q_3 = 0$. In other words, the mapping $x \to q$ is conformal at $\partial\Omega \cup U$. Next denote by $y(q)$ the conformal mapping of the half-space $\{q_3 > 0\}$ onto the unit ball B such that $y(0) = P^+$. It is clear that the composite mapping $y \to q \to x$ meets all requirements of the lemma. $\square$

Now fix an arbitrary point $P \in \partial\Omega$ and diffeomorphism $y : U \to B^+$ satisfying all hypotheses of Lemma 10.6. Set

$$\overline{u}(y) =: u(x(y)), \quad \overline{\phi}(y) =: \phi(x(y)), \quad \overline{\mathbf{W}} = \mathbf{W}(x(y)).$$

Straightforward calculations lead to the identity

$$[\operatorname{div} \mathbf{W}](x(y)) = \operatorname{div} \mathbf{V}(y) + \mathbf{b}(y) \cdot \mathbf{V}(y) \text{ in } B^+ \cap B, \tag{10.18}$$

where

$$\mathbf{V} = \mathbb{Y}^* \overline{\mathbf{W}}, \quad \mathbf{b} = (b_1, b_2, b_3) \quad b_j(y) = \partial^2_{y_k y_j} x_i(y)\, \partial_{x_i} y_k(y).$$

Now our aim is to derive equations for the vector field $\mathbf{V}$. We begin with the observation that the normal vector $\mathbf{n}$ to $\partial\Omega$ and the normal vector $\overline{\nu} = y$ to ∂B are related by the formula

$$\mathbf{n}(x(y)) = |\mathbb{Y}y|^{-1}\mathbb{Y}y = g^{-1/6}\mathbb{Y}y \text{ on } \partial B \cap B^+.$$

On the other hand, we have $\nabla\phi = \mathbb{Y}\nabla\overline{\phi}$ in $B^+ \cap B$, and $\mathbb{Y}^*\mathbb{Y} = g^{1/3}\mathbf{I}$ on $\partial B \cap B^+$. Thus, we get

$$\partial_n \phi\, \mathbf{n}(x(y)) = (\nabla\overline{\phi} \cdot y)\mathbb{Y}y = \partial_\nu \overline{\phi}\, \mathbb{Y}\, \overline{\nu} \text{ on } \partial B \cap B^+. \tag{10.19}$$

The change of variables in (10.16) leads to the following equations for the vector field $\overline{\mathbf{W}}$ and the function $\overline{\phi}$

$$\mathscr{B}\overline{\mathbf{W}} = 0, \quad \mathscr{B}\overline{\phi} = \overline{u} \text{ in } B \cup B^+,$$

where the Beltrami operator is defined by the equality

$$\mathscr{B}u =: g^{1/2}\operatorname{div}\big(g^{-1/2}\mathbf{g}\nabla u\big).$$

¿From this, the expression for $\mathbf{V}$, and identity (10.19) we obtain the following equations for $\mathbf{V} = (V_1, V_2, V_3)$ and $\overline{\phi}$,

$$\begin{aligned}\mathscr{B}\mathbf{V}_j = \mathcal{N}(\overline{W}_k, Y_{jk}) \text{ in } B\cap B^+, \quad \mathbf{V} = \partial_\nu(g^{1/3}\overline{\phi})\,\overline{\nu} \text{ on } \partial B\cap B^+,\\ \mathscr{B}\overline{\phi} = \overline{u} \text{ in } B\cap B^+, \quad \overline{\phi} = 0 \text{ on } \partial B\cap B^+.\end{aligned} \tag{10.20}$$

Here the bilinear form $\mathcal{N}$ is defined by

$$\mathcal{N}(f,h) = f\mathscr{B}(h) + 2\mathbf{g}\nabla f\cdot\nabla h = 2\operatorname{div}(f\mathbf{g}\nabla h) + a(\mathscr{B}(h) - 2\operatorname{div}(\mathbf{g}\nabla h)).$$

Step 2. The next step is the localization of all introduced functions inside the ball B^+. Note that the fundamental matrix $\mathbf{g}$ can be extended over $B\backslash B^+$ such that the extension (also denoted by $\mathbf{g}$) is positive, infinitely differentiable, and coincides with $g^{1/3}\mathbf{I}$ on ∂B. In other words, $\mathbf{g} = g^{1/3}\mathbf{I} + (1-|y|^2)\mathbb{G}$ with $\mathbb{G}\in C^\infty(B)$. Next choose an arbitrary function $\eta\in C_0^\infty(B^+)$ and set

$$v = \eta\overline{u}, \quad \psi = g^{1/3}\eta\overline{\phi}, \quad \mathbf{H} = \eta\,\mathbf{V}.$$

Assume that the vector field $\mathbf{H}$ and functions v, ψ are extended by 0 over $B\backslash B^+$. It follows from (10.20) and the identity $\eta\mathscr{B}(f) = \mathscr{B}(\eta f) - \mathcal{N}(f,\eta)$ that the extended vector field $\mathbf{H} = (H_1, H_2, H_3)$ and the functions ψ, v_0 satisfy the following equations and boundary conditions

$$\mathscr{B}H_j = \mathcal{N}(V_j,\eta) + \eta\mathcal{N}(\overline{W}_k, Y_{jk}) \text{ in } B, \quad \mathbf{H} = \partial_\nu\psi\,\overline{\nu} \text{ on } \partial B, \tag{10.21a}$$

$$\mathscr{B}\psi = g^{1/3}v + \mathcal{N}(\overline{\phi}, g^{1/3}\eta) \text{ in } B, \quad \psi = 0 \text{ on } \partial B, \tag{10.21b}$$

$$\mathscr{B}v = \mathcal{N}(\overline{u},\eta) \text{ in } B, \quad v = \eta\overline{u} \text{ on } \partial B. \tag{10.21c}$$

In these equations all quantities are compactly supported in $B^+\cap B$ and extended by 0 over $B\setminus B^+$. Split $\mathbf{H}$ and ψ into the parts

$$\mathbf{H} = \mathbf{H}_0 + \mathbf{H}_1, \quad \psi = \psi_0 + \psi_1, \tag{10.22}$$

where $\mathbf{H}_0$ and ψ_0 are solutions to the boundary value problems

$$\begin{aligned}\Delta\mathbf{H}_0 = 0 \text{ in } B, \quad \mathbf{H}_0 = \partial_\nu\psi_0\,\overline{\nu} \text{ on } \partial B,\\ \Delta\psi_0 = v_0 \text{ in } B, \quad \psi_0 = 0 \text{ on } \partial B,\\ \Delta v_0 = 0 \text{ in } B, \quad v_0 = v \text{ on } \partial B.\end{aligned} \tag{10.23}$$

It follows from this and (10.21) that the reminders $\mathbf{H}_1$ and ψ_1 satisfy the equations

$$\begin{aligned}\Delta H_{1,j} = g^{-1/3}\big(\mathcal{N}(V_j,\eta) + \eta\mathcal{N}(\overline{W}_k, Y_{jk}) - \mathscr{B}_1 H_j\big) \text{ in } B,\\ \mathbf{H}_1 = \partial_\nu\psi_1\,\overline{\nu} \text{ on } \partial B,\end{aligned} \tag{10.24a}$$

$$\Delta\psi_1 = v - v_0 + g^{-1/3}\big(\mathcal{N}(g^{-1/3}\eta,\overline{\phi}) - \mathscr{B}_1\psi\big) \text{ in } B, \psi_1 = 0 \text{ on } \partial B, \tag{10.24b}$$

in which the second order differential operator $\mathscr{B}_1 =: \mathscr{B} - g^{1/3}\Delta$ is defined by the equality

$$g^{-1/2}\mathscr{B}_1 v = \operatorname{div}\Big(g^{-1/2}\mathbb{G}\,\nabla\big((1-|y|^2)v\big) + 2v\,\mathbb{G}\,y\Big) + \nabla g^{-1/6}\nabla v.$$

Identity (10.18) implies the equality

$$\eta(y)[\operatorname{div}\mathbf{W}](x(y)) = \operatorname{div}\mathbf{H}(y) - \nabla\eta\cdot\mathbf{V}(y) + \mathbf{b}(y)\cdot\mathbf{H}(y) \text{ in } B^+\cap B,$$

which along with (10.15) yields the identity

$$\eta(y)[\mathscr{A}u - u](x(y)) = -\operatorname{div}\mathbf{H}(y) + \nabla\eta\cdot\mathbf{V}(y) - \mathbf{b}(y)\cdot\mathbf{H}(y) \text{ in } B^+\cap B.$$

The next step is crucial for our approach. It is easy to see that by virtue of (10.23), the vector field $\mathbf{H}_0$ and the functions ψ_0, v_0 meet all requirements of Theorem 10.1, which gives

$$\operatorname{div}\mathbf{H}_0(y) = \frac{1}{2}v_0 + \mathscr{V}v_0 = \frac{1}{2}\eta\overline{u} + \frac{1}{2}(v_0 - v) + \mathscr{V}v_0.$$

Recalling the identity $\overline{u}(y) = u(x(y))$ we finally obtain

$$\eta(y)[\mathscr{A}u - \frac{1}{2}u](x(y)) = \mathscr{E}_\eta u, \tag{10.25}$$

where

$$\mathscr{E}_\eta u = -\frac{1}{2}(v_0 - v) - \mathscr{V}v_0 - \operatorname{div}\mathbf{H}_1(y) + \nabla\eta\mathbf{V}(y) - \mathbf{b}(y)\cdot\mathbf{H}(y) \text{ in } B^+\cap B. \tag{10.26}$$

Notice that the function $\mathscr{E}_\eta u\in C^\infty(B)$ is compactly supported in $B\cap B^+$.

Step 3. Now our aim is to estimate $\mathscr{E}_\eta$ in the Sobolev spaces. This procedure is based on the following lemma.

Lemma 10.7. *Let Let $\mathbf{g}\in(C^\infty(B))^9$ be a positive symmetric matrix field such that*

$$\mathbf{g} = g^{1/3}\mathbf{I} + (1-|y|^2)\mathbb{G}, \quad \mathbf{G}\in C^\infty(B).$$

Furthermore, assume that functions $v, f\in C^\infty(B)$ satisfy the equation

$$\mathscr{B}v = f \text{ in } B,$$

and v_0 is a solution to the Dirichlet problem

$$\Delta v_0 = 0 \text{ in } B, \quad v_0 = v \text{ on } \partial B.$$

Then for any $r\in(1,\infty)$ and integer $s\geq 0$,

$$\|(1-|y|^2)v\|_{H^{s+1,r}(B)} + \|\mathscr{B}_1 v\|_{H^{s-1,r}(B)} \leq c(\|f\|_{H^{s-1,r}(B)} + \|v\|_{H^{s,r}(B)}), \tag{10.27}$$

$$\|v_0\|_{H^{s,r}(B)} + \|v - v_0\|_{H^{s+1,r}(B)} \leq c(\|f\|_{H^{s-1,r}(B)} + \|v\|_{H^{s,r}(B)}), \tag{10.28}$$

where the constant c depends only on $r, s,$ and $\mathbf{g}$.

Proof. We begin with the observation that the function $(1-|y|^2)v$ satisfies the equation and the boundary conditions

$$\mathscr{B}((1-|y|^2)v) = (1-|y|^2)f + \mathscr{N}(v, 1-|y|^2) \text{ in } B,$$
$$(1-|y|^2)v = 0 \text{ on } \partial B.$$

It follows from Lemma 1.1 that

$$\|(1-|y|^2)v\|_{H^{s+1,r}(B)} \leq c(\|f\|_{H^{s-1,r}(B)} + \|\mathscr{N}(v, 1-|y|^2)\|_{H^{s-1,r}(B)}).$$

On the other hand, we have

$$\|\mathscr{N}(v, 1-|y|^2)\|_{H^{s-1,r}(B)} \leq c\|v\|_{H^{s,r}(B)},$$

which leads to the estimate

$$\|(1-|y|^2)v\|_{H^{s+1,r}(B)} \leq c(\|f\|_{H^{s-1,r}(B)} + \|v\|_{H^{s,r}(B)}).$$

¿From this and the expression for the differential operator $\mathscr{B}_1$ we obtain estimate (10.27). Next note that the function $v-v_0$ satisfies the equations and the boundary conditions

$$\Delta(v-v_0) = g^{-1/3}(f-\mathscr{B}_1 v) \text{ in } B, \quad v-v_0 = 0 \text{ on } \partial B.$$

Lemma 1.1 along with inequality (10.27) for the right-hand side imply the estimate

$$\|v-v_0\|_{H^{s+1,r}(B)} \le c(\|f\|_{H^{s-1,r}(B)} + \|v\|_{H^{s,r}(B)}).$$

which completes the proof. □

The next lemma gives the key estimate for $\mathscr{E}_\eta$.

Lemma 10.8. *Under the above assumptions, there exists a constant c depending only on exponents $r \in (1,\infty)$, $s \ge 0$, the function η and the fundamental matrix $\mathbf{g}$ such that the inequality*

$$\|\mathscr{E}_\eta u\|_{H^{s+1,r}(B)} \le c(r,s,\eta,\mathbf{g})\|u\|_{H^{s,r}(\Omega)}$$

holds true for all smooth harmonic functions u.

Proof. We estimate step by step all quantities in the expression for $\mathcal{E}_\eta u$ starting with v. Since $v = \eta\overline{u}$, we have

$$\|v\|_{H^{s,r}(B)} \le c\|\overline{u}\|_{H^{s,r}(B\cap B^+)} \le c\|u\|_{H^{s,r}(\Omega)}, \tag{10.29}$$

which leads to the estimate

$$\|\mathcal{N}(\overline{u},\eta)\|_{H^{s-1,r}(B)} \le c\|\overline{u}\|_{H^{s,r}(B\cap B^+)} \le c\|u\|_{H^{s,r}(\Omega)}.$$

The function v meets all requirements of Lemma 10.7 with $f = \mathcal{N}(\overline{u}\eta)$. Applying this lemma and using inequality (10.29) we obtain the following estimates for v and v_0

$$\|(1-|y|^2)v\|_{H^{s+1,r}(\Omega)} + \|v_0\|_{H^{s,r}(B)} + \|v-v_0\|_{H^{s+1,r}(B)} \le c\|u\|_{H^{s,r}(B)}. \tag{10.30}$$

Next applying Theorem 10.1 to the harmonic function v_0 we arrive at the estimate

$$\|\mathscr{V} v_0\|_{H^{s+1,r}(B)} \le c\|u\|_{H^{s,r}(B)}. \tag{10.31}$$

Now our task is to estimate $\mathbf{V}$ and $\mathbf{H}$. Applying Lemma 1.1 to boundary value problem (10.16) we get the inequality

$$\|\mathbf{W}\|_{H^{s+1,r}(\Omega)} \le c\|\phi\|_{H^{s+2,r}(\Omega)} \le c\|u\|_{H^{s,r}(\Omega)}.$$

Since η is compactly supported in the ball B^+, we conclude from this that

$$\|\psi\|_{H^{s+2,r}(B)} \le c\|\overline{\phi}\|_{H^{s+2,r}(B\cap B^+)} \le c\|\phi\|_{H^{s+2,r}(\Omega)} \le c\|u\|_{H^{s,r}(\Omega)}, \tag{10.32}$$

$$\|\mathbf{H}\|_{H^{s+1,r}(B)} \le c\|\mathbf{V}\|_{H^{s+1,r}(B\cap B^+)} \le c\|\overline{\mathbf{W}}\|_{H^{s+1,r}(B\cap B^+)} \le c\|u\|_{H^{s,r}(\Omega)} \ . \tag{10.33}$$

Next we derive estimates for the auxiliary function ψ_1. It follows from (10.21b) that the function $(1-|y|^2)\psi$ satisfies the equation

$$\mathscr{B}((1-|y|^2)\psi) = (1-|y|^2)(g^{1/3}v + \mathcal{N}(\overline{\phi}, g^{1/3}\eta)) + \mathcal{N}(\psi,(1-|y|^2)), \tag{10.34}$$

and vanishes on $\partial\Omega$. On the other hand, inequalities (10.29) and (10.32) imply the following estimate for that the right- hand side of this equation

$$\begin{aligned}\|(1-|y|^2)(g^{1/3}v + \mathcal{N}(\overline{\phi}, g^{1/3}\eta)) + \mathcal{N}(\psi,(1-|y|^2))\|_{H^{s+1,r}(\Omega)} \\ \le c(\|u\|_{H^{s,r}(\Omega)} + \|\overline{\phi}\|_{H^{s+2,r}(B\cap B^+)} + \|\psi\|_{H^{s+2,r}(B)}) \le c\|u\|_{H^{s,r}(\Omega)}.\end{aligned}$$

Applying Lemma 1.1 to equation (10.34) we obtain that $\|(1-|y|^2)\psi\|_{H^{s+3,r}(\Omega)} \le c\|u\|_{H^{s,r}(\Omega)}$, which gives

$$\|\mathscr{B}_1(\psi)\|_{H^{s+1,r}(\Omega)} \le c\|\psi\|_{H^{s+2,r}(\Omega)} + c\|(1-|y|^2)\psi\|_{H^{s+3,r}(\Omega)} \le c\|u\|_{H^{s,r}(\Omega)}. \tag{10.35}$$

This result along with inequalities (10.30) gives the following estimate for the right-hand side of equation (10.24b),

$$\|v - v_0 + g^{-1/3}\big(\mathcal{N}(\overline{\phi}, g^{-1/3}\eta) - \mathscr{B}_1\psi\big)\|_{H^{s+1,r}(\Omega)} \le$$
$$c\|u\|_{H^{s,r}(\Omega)} + c\|\overline{\phi}\|_{H^{s+2,r}(B\cap B^+)} \le c\|u\|_{H^{s,r}(\Omega)}.$$

Applying Lemma 1.1 to equation (10.24b) we derive the estimate for ψ_1,

$$\|\psi_1\|_{H^{s+3,r}(\Omega)} \le c\|u\|_{H^{s,r}(\Omega)}. \tag{10.36}$$

Now we can estimate the vector field $\mathbf{H}_1$. Applying Lemma 10.7 to equation (10.21a) and using estimates (10.33) we obtain

$$\|\mathscr{B}_1(H_j)\|_{H^{s,r}(\Omega)} \le c\|\mathcal{N}(V_j,\eta) + \eta\mathcal{N}(\overline{W}_k, Y_{jk})\|_{H^{s,r}(\Omega)} \le c\|u\|_{H^{s,r}(\Omega)} .$$

It follows from this and (10.33) that the right- hand side of equation (10.24a) satisfies the inequality

$$\|\mathcal{N}(V_j,\eta) + \eta\mathcal{N}(\overline{W}_k, Y_{jk}) - \mathscr{B}_1(H_j)\|_{H^{s,r}(\Omega)} \le c\|u\|_{H^{s,r}(\Omega)}.$$

Applying Lemma 1.1 to equations (10.24a) and using (10.36) we get

$$\|\mathbf{H}_1\|_{H^{s+2,r}(\Omega)} \le c\|u\|_{H^{s,r}(\Omega)} + c\|\nabla\psi\cdot y\|_{H^{s+2,r}(\Omega)} \le c\|u\|_{H^{s,r}(\Omega)}.$$

Combining this result with inequalities (10.30), (10.31), and (10.33) we conclude that $\|\mathcal{E}_\eta u\|_{H^{s+1,r}(\Omega)} \le c\|u\|_{H^{s,r}(\Omega)}$, which completes the proof. □

Step 3. Since $\partial\Omega$ is compact, there is a finite collection of points $P_i \in \partial\Omega$, $1 \le i \le n$, such that the corresponding neighborhoods U_i given by Lemma 10.6 cover $\partial\Omega$. Denote by K the compact set $\Omega\setminus\bigcup U_i$ and set $d =$ dist. $(K,\partial\Omega)$. There is a collection of points $P_i \in K$, $n+1 \le i \le m$, such that the balls $U_i = B(P_i, d/2)$ cover K. Obviously the sets U_i, $1 \le i \le m$, cover Ω. It is well known that there are smooth functions ζ_i, $1 \le i \le m$, with the properties

$$\zeta_i \in C_0^\infty(\mathbb{R}^3), \quad \text{spt } \zeta_i \Subset U_i, \quad \sum_i \zeta_i = 1.$$

We have for any $u \in C^\infty(\Omega)$,

$$\mathscr{A}u - u/2 = \sum_i \zeta_i(\mathscr{A}u - u/2).$$

For each $1 \le i \le n$, denote by η_i the function $\eta_i(y) = \zeta(\mathbf{x}(y))$, where $\mathbf{x}(y)$ is a diffeomorphism of the ball B^+ onto U_i defined by Lemma 10.6. It follows from this lemma that $\eta_i \in C_0^\infty(\mathbb{R}^3)$, and spt $\eta \Subset B^+$. In particular, we have $\|\zeta_i(\mathscr{A}u - u/2)\|_{H^{s,r}(\Omega)} \le \|\mathcal{E}_{\eta_i}u\|_{H^{s,r}(B)}$, where $\mathcal{E}_\eta$ is given by formula (10.26). From this, identity (10.25) and Lemma 10.8 we conclude that

$$\|\zeta_i(\mathscr{A}u - u/2)\|_{H^{1+s,r}(\Omega)} \le c(s,r,\Omega)\|u\|_{H^{s,r}(\Omega)} \text{ for } 1 \le i \le n. \tag{10.37}$$

On the other hand, the expression $\mathscr{A} = \operatorname{div}\Delta^{-1}\nabla$ yields the identity

$$\zeta_i(\mathscr{A}u - u/2) = \operatorname{div}(\zeta_i\mathbf{w}) - \nabla\zeta_i\cdot\mathbf{w} - \zeta_i u/2,$$

where $\mathbf{w}$ is a solution to the boundary value problem

$$\Delta\mathbf{w} = \nabla u \text{ in } \Omega, \quad \mathbf{w} = 0 \text{ on } \partial\Omega.$$

We have

$$\Delta(\zeta_i \mathbf{w}) = 2\,\mathrm{div}(\nabla\zeta_i \otimes \mathbf{w}) + \Delta\zeta_i \mathbf{w} + \nabla(\zeta_i u) - u\nabla\zeta_i \text{ in } \Omega,$$
$$\Delta(\zeta_i u) = 2\,\mathrm{div}(u\nabla\zeta_i) - \Delta\zeta_i u \text{ in } \Omega.$$

Since functions $\zeta_i \mathbf{w}$ and $\zeta_i u$ vanish on $\partial\Omega$ for all $i \geq 1$, we conclude from this and Lemma 4.1 that inequalities

$$\|\mathbf{w}\|_{H^{s+1,r}(\Omega)} \leq c\|u\|_{H^{s,r}(\Omega)}, \quad \|\zeta_i u\|_{H^{s+1,r}(\Omega)} \leq c\|u\|_{H^{s,r}(\Omega)},$$
$$\|\zeta_i \mathbf{w}\|_{H^{s+2,r}(\Omega)} \leq c(\|\mathbf{w}\|_{H^{s+1,r}(\Omega)} + \|u\|_{H^{s,r}(\Omega)}) \leq c\|u\|_{H^{s,r}(\Omega)},$$

hold true for all $i \geq n+1$. Therefore, $\|\zeta_i(\mathscr{A}u - u/2)\|_{H^{s+1,r}(\Omega)} \leq c\|u\|_{H^{s,r}(\Omega)}$ for $n+1 \leq i \leq m$. This estimate along with inequalities (10.37) implies estimate (10.14), and the theorem follows. □

10.2. **Proof of Theorem 1.9.** We are now in a position to complete the proof of Theorem1.9. Denote by $b^{s,r}(\Omega)$ the set of all harmonic functions $u \in H^{s,r}(\Omega)$, and by $b_c(\Omega)$ the set of all harmonic functions of class $C^\infty(\Omega)$. We will write simply b^r instead of $b^{0,r}$ It is clear that $b^{s,r}(\Omega)$ is a closed subspace of $H^{s,r}(\Omega)$. Let us prove that $b_c(\Omega)$ is dense in $b^{s,r}(\Omega)$ for all $1 < r < \infty$ and integer $s \geq 0$. Assume $s \geq 1$, and choose an arbitrary $u \in b^{s,r}(\Omega)$. Then there is a sequence $v_n \in C^\infty(\Omega)$ such that $v_n \to u$ in $H^{s,r}(\Omega)$ as $n \to \infty$. Denote by $u_n \in C^\infty(\Omega)$ a solution to the boundary value problem $\Delta u_n = 0$ in Ω and $u_n = v_n$ at $\partial\Omega$. It remains to note that by Lemma 1.1, $\|u - u_n\|_{H^{s,r}(\Omega)} \leq c\|u - v_n\|_{H^{s,r}(\Omega)} \to 0$ as $n \to \infty$.
It remains to prove that $b_c(\Omega)$ is dense in $b^r(\Omega)$. Suppose, in contrary to our claim, that there is a function $u \in b^r(\Omega) \setminus \mathrm{cl}\, b_c(\Omega)$. The Hahn-Banach theorem yields the existence of $w \in L^{r'}(\Omega)$ such that

$$\langle \zeta, w\rangle = 0 \text{ for all } \zeta \in b_c(\Omega) \text{ and } \langle u, w\rangle \neq 0\ . \tag{10.38}$$

Set $\varphi = \Delta^{-1}(w) \in H^{2,r'}(\Omega) \cap H_0^{1,r'}(\Omega)$, and choose an arbitrary function $v \in C^\infty(\partial\Omega)$. Denote by $\zeta \in b_c(\Omega)$ the harmonic extension of v onto Ω. Since $\langle \zeta, w\rangle = 0$, the Green formula implies

$$0 = \langle \zeta, \Delta\varphi\rangle - \langle \varphi, \Delta\eta\rangle = \int\limits_{\partial\Omega} v\partial_n\varphi\, ds, \quad \forall v \in C_0^\infty(\Omega),$$

which yields the equality $\partial_n\varphi = 0$ and $\varphi \in H_0^{2,r'}(\Omega)$. Hence there exists a sequence of functions $\varphi_n \in C_0^\infty(\Omega)$ such that $\varphi_n \to \varphi$ in $H^{2,r'}(\Omega)$ and $\Delta\varphi_n \to w$ in $L^{r'}(\Omega)$ as $n \to \infty$. From this we conclude that $0 = \langle u, \Delta\varphi_n\rangle \to \langle u, w\rangle$, which contradicts to (10.38).
Next notice that by virtue of Lemma 4.1, operator $\mathscr{A} : H^{s,r}(\Omega) \to H^{s,r}(\Omega)$ is bounded for all $s \geq 0$ and $r \in (1, \infty)$. In particular, $\mathscr{A}u - u/2$ is well defined for all $u \in H^{s,r}(\Omega)$. Since $b_c(\Omega)$ is dense in $b^{s,r}(\Omega)$, this result along with Theorem 10.5 yields the following.

Corollary 10.9. *Under the assumptions of Theorem 10.5, inequality* (10.14) *holds true for all harmonic functions* $u \in H^{s,r}(\Omega)$.

Now we can derive the representation (*ii*) in Theorem 10.5. Denote by $o_c(\Omega) \subset C^\infty(\Omega)$ the linear space of all functions $\zeta = \Delta\varphi$ with an arbitrary $\varphi \in C_0^\infty(\Omega)$. We shall consider $o_c(\Omega)$ as a linear subspace of $L^2(\Omega)$. It is clear that a function

$u \in L^2(\Omega)$ is harmonic if and only if $\langle u, \zeta \rangle = 0$ for all $\zeta \in o_c(\Omega)$. In other words, $\mathrm{cl}\, o_c(\Omega) = b^2(\Omega)^\perp$. For any element $\zeta \in o_c(\Omega)$, we have

$$\mathscr{A}\zeta = \operatorname{div} \Delta^{-1} \nabla \Delta \varphi = \operatorname{div} \Delta^{-1} \Delta \nabla \varphi = \Delta \varphi = \zeta.$$

Since the operator $\mathscr{A}$ is bounded in $L^2(\Omega)$, it follows from this that that $\mathbf{A}\zeta = \zeta$ for all $\zeta \in \mathrm{cl}\, o_c(\Omega) = b^2(\Omega)^\perp$, and hence $\mathscr{A}(\mathbf{I} - \mathscr{Q}) = \mathbf{I} - \mathscr{Q}$. Thus, we arrive to the desired representation

$$\mathscr{A} u \equiv u - \frac{1}{2}\mathscr{Q} u + \mathscr{K} u, \text{ where } \mathscr{K} = (\mathscr{A} - \frac{1}{2}\mathbf{I})\mathscr{Q}. \tag{10.39}$$

Next we prove that the projection $\mathscr{Q}$ is bounded in $L^r(\Omega)$. For $r \in [2, 6]$, we have

$$\|\mathscr{Q} u\|_{L^r(\Omega)} \le 2\|\mathscr{A} u\|_{L^r(\Omega)} + \|u\|_{L^r(\Omega)} + 2\|\mathscr{K} u\|_{L^6(\Omega)} \le c\|u\|_{L^r(\Omega)} + c\|\mathscr{K} u\|_{H^{1,2}(\Omega)}.$$

On the other hand, Corollary 10.9 yields the estimate

$$\|\mathscr{K} u\|_{H^{1,2}(\Omega)} = \|(\mathscr{A} - \mathbf{I}/2)\mathscr{Q} u\|_{H^{1,2}(\Omega)} \le c\|\mathscr{Q} u\|_{L^2(\Omega)} \le c\|u\|_{L^2(\Omega)}.$$

Combining these estimates we conclude that the operator $\mathscr{Q} : L^r(\Omega) \mapsto L^r(\Omega)$ is bounded for all $r \in [2, 6]$. Noting that the embedding $H^{1,6}(\Omega) \hookrightarrow L^r(\Omega)$, $r > 1$, is bounded and arguing as before we obtain that for all $r > 6$,

$$\|\mathscr{Q} u\| \le c\|u\|_{L^r(\Omega)} + c\|\mathscr{K} u\|_{H^{1,6}(\Omega)} \le c\|u\|_{L^r(\Omega)} + c\|\mathscr{Q} u\|_{L^6(\Omega)} \le c\|u\|_{L^r(\Omega)}.$$

Hence the projection $\mathscr{Q}$ is bounded in $L^r(\Omega)$ for all $r \ge 2$. Since the projection is symmetric, the boundedness of Q in $L^r(\Omega)$ for $r \in (1, 2)$ follows from the duality argument. Hence the inequality $\|\mathscr{Q} u\|_{H^{s,r}(\Omega)} \le c\|u\|_{H^{s,r}(\Omega)}$ is fulfilled for $s = 0$. Assuming that this inequality holds for s, we will prove it for $s + 1$. To this end note that, by virtue of Corollary 10.9,

$$\begin{aligned} \|\mathscr{Q} u\|_{H^{s+1,r}(\Omega)} &\le 2\|\mathscr{A} u\|_{H^{s+1,r}(\Omega)} + \|u\|_{H^{s+1,r}(\Omega)} + 2\|\mathscr{K} u\|_{H^{s+1,r}(\Omega)} \\ &\le c\|u\|_{H^{s+1,r}(\Omega)} + c\|\mathscr{Q} u\|_{H^{s,r}(\Omega)} \le c\|u\|_{H^{s+1,r}(\Omega)}. \end{aligned}$$

Therefore, the operator $\mathscr{Q}$ is bounded in $H^{s+1,r}(\Omega)$, and hence it is bounded in $H^{s,r}(\Omega)$ for any $r \in (0, \infty)$ and integer $s \ge 0$. From this and Corollary 10.9 we conclude that the operator $\mathscr{K} : H^{s,r}(\Omega) \mapsto H^{s,r}(\Omega) \to H^{s+1,r}(\Omega)$ is bounded for all $r > 1$ and integer $s \ge$. It remains to note that the boundedness $\mathscr{Q}$ and $\mathscr{K}$ for real $s \ge 0$ follows from the interpolation theory, which completes the proof. □

Appendix A. Proof of Lemmas 9.2 and 9.3

Proof of Lemma 9.2 We start with the proof of (**P**1). It follows from emergent field Condition 1.4 that for for each point $P \in \Gamma$, there exist the standard Cartesian coordinates (x_1, x_2, x_3) with the origin at P such that in the new coordinates $\mathbf{U}(P) = (U, 0, 0)$ with $U = |\mathbf{U}(P)|$, and $\mathbf{n}(P) = (0, 0, -1)$. Moreover, there is a neighborhood $\mathcal{O} = [-k, k]^2 \times [-t, t]$ of P such that the intersections $\partial\Omega \cap \mathcal{O}$ and $\Gamma \cap \mathcal{O}$ are defined by the equations

$$F_0(x) \equiv x_3 - F(x_1, x_2) = 0, \quad \nabla F_0(x) \cdot \mathbf{U}(x) = 0,$$

and $\Omega \cap \mathcal{O}$ is the epigraph $\{F_0 > 0\} \cap \mathcal{O}$. The function F satisfies the conditions

$$\|F\|_{C^2([-k,k]^2)} \le K, \quad F(0, 0) = 0, \quad \nabla F(0, 0) = 0, \tag{11.1}$$

where the constants $k, t < 1$ and $K > 1$ depend only on the curvature of $\partial\Omega$ and are independent of the point P. Since the vector field $\mathbf{U}$ is transversal to Γ, the manifold $\Gamma \cap \mathcal{O}$ admits the parameterization

$$x = \big(\mathfrak{g}(y_2),\, y_2,\, F(\mathfrak{g}(y_2), y_2)\big), \tag{11.2}$$

such that $\mathfrak{g}(0) = 0$ and $\|\mathfrak{g}\|_{C^2([-k,k])} \le C$, where the constant $C > 1$ depends only on Ω and $\mathbf{U}$.

With this notation, the inequality (1.10) implies the existence of positive constants $N^{\pm}$ independent of P such that for $x \in \partial\Omega$ given by the condition $F_0(x_1, x_2, x_3) = x_3 - F(x_1, x_2) = 0$, we have

$$\begin{gathered} N^-\big(x_1 - \mathfrak{g}(x_2)\big) \le -\nabla F_0(x)\cdot \mathbf{U}(x) \le N^+\big(x_1 - \mathfrak{g}(x_2)\big) \text{ for } x_1 > \mathfrak{g}(x_2),\\ -N^-\big(x_1 - \mathfrak{g}(x_2)\big) \le \nabla F_0(x)\cdot \mathbf{U}(x) \le -N^+\big(x_1 - \mathfrak{g}(x_2)\big) \text{ for } x_1 < \mathfrak{g}(x_2). \end{gathered} \tag{11.3}$$

Choose the standard Cartesian coordinate system (x_1, x_2, x_3) associated with the point P. Let us consider the Cauchy problem.

$$\begin{aligned} \partial_{y_1}\mathbf{x} = \mathbf{u}(\mathbf{x}(y)) \text{ in } Q_a,\\ x_1(y) = \mathfrak{g}(y_2), \quad x_2(y) = y_2 \text{ for } y_1 = 0,\\ x_3 = F(\mathfrak{g}(y_2), y_2) + y_3 \text{ for } y_1 = 0. \end{aligned} \tag{11.4}$$

Without any loss of generality we can assume that $0 < a < k < 1$. For any such a, problem (11.4) has a unique solution of class $C^1(Q_a)$. Denote by $\mathfrak{F}(y) = D_y\mathbf{x}(y)$. The calculations show that

$$\mathfrak{F}_0 := \mathfrak{F}(y)\Big|_{y_1=0} = \begin{pmatrix} u_1 & \mathfrak{g}'(y_2) & 0\\ u_2 & 1 & 0\\ u_3 & \partial_{y_2}F(\mathfrak{g}(y_2), y_2) & 1 \end{pmatrix}, \mathfrak{F}(0) = \begin{pmatrix} U & \mathfrak{g}'(0) & 0\\ 0 & 1 & 0\\ 0 & 0 & 1 \end{pmatrix},$$

which implies

$$\|\mathfrak{F}(0)^{\pm 1}\| \le C/3, \quad \|\mathfrak{F}_0(y) - \mathfrak{F}(0)\| \le ca. \tag{11.5}$$

Differentiation of (11.4) leads to the ordinary differential equation for $\mathfrak{F}$

$$\partial_{y_1}\mathfrak{F} = D_y\mathbf{u}(\mathbf{x})\mathfrak{F}, \quad \mathfrak{F}\Big|_{y_1=0} = \mathfrak{F}_0.$$

¿From this get

$$\partial_{y_1}\|\mathfrak{F} - \mathfrak{F}_0\| \le M(\|\mathfrak{F} - \mathfrak{F}_0\| + \|\mathfrak{F}_0\|),$$

and hence $\|\mathfrak{F} - \mathfrak{F}_0\| \le c(M)\|\mathfrak{F}_0\|a$. Combining this result with (11.5) we finally arrive at

$$\|\mathfrak{F}(y) - \mathfrak{F}(0)\| \le ca. \tag{11.6}$$

This inequality along with the implicit function theorem implies the existence of $a > 0$, depending only on M and Ω, such that the mapping $x = \mathbf{x}(y)$ takes diffeomorphically the cube Q_a onto some neighborhood of the point P and satisfies inequalities (9.4).

Let us turn to the proof of $(\mathbf{P}2)$. We begin with the observation that the manifold $\mathbf{x}^{-1}(\partial\Omega \cap \mathcal{O})$ is defined by the equation

$$\Phi_0(y) := x_3(y) - F(x_1(y), x_2(y)) = 0, \quad y \in Q_a.$$

Let us show that Φ_0 is strictly monotone in y_3 and has the opposite signs on the faces $y_3 = \pm a$. To this end note that the formula for $\mathfrak{F}(0)$ along with (11.6) implies the estimates

$$|\partial_{y_3} x_3(y) - 1| + |\partial_{y_3} x_1(y)| + |\partial_{y_3} x_2(y)| \le ca \text{ in } Q_a.$$

Thus, we get

$$1 - ca \le \partial_{y_3}\Phi_0(y) = \partial_{y_3} x_3(y) - \partial_{x_i} F(x_1, x_2)\partial_{y_3} x_i(y) \le 1 + ca.$$

It follows from (11.6), that for $y_3 = 0$, we have $|x_3(y)| \le ca|y|$, which along with (9.4) yields the estimate

$$|\Phi_0(y)| \le |x_3(y)| + |F(x(y))| \le ca|y| + KC|y|^2 \le ca^2 \text{ for } y_3 = 0.$$

Hence there is a positive a depending only on M and Ω, such that the inequalities

$$1/2 \le \partial_{y_3}\Phi_0(y) \le 2, \quad \pm\Phi_0(y_1, y_2, \pm a) > 0,$$

hold true for all $y \in Q_a$. Therefore, the equation $\Phi_0(y) = 0$ has a unique solution $y_3 = \Phi(y_1, y_2)$ in the cube Q_a. Moreover the function $\Phi \in C^1([-a, a]^2)$ vanishes for $y_1 = y_3 = 0$. Thus, we get

$$\mathcal{P}_a := \mathbf{x}^{-1}(\mathcal{O} \cap \Omega) = \{\Phi(y_1, y_2) < y_3 < a, \quad |y_1|, |y_2| \le a\}.$$

Note that $|\mathbf{u}(\mathbf{x}(y)) - U\mathbf{e}_1| \le M|\mathbf{x}(y)| \le Ca$. Therefore, we can choose $a = a(M, \Omega)$ such that $2U/3 \le u_1 \le 4U/3$ and $C|u_2| \le U/3$ in Q_a. Recall that $x_1(y) - \mathfrak{g}(x_2(y))$ vanishes at the plane $y_1 = 0$ and

$$\partial_{y_1}\big[x_1(y) - \mathfrak{g}(x_2(y))\big] = u_1(y) - \mathfrak{g}'(x_2(y))u_2(y).$$

We obtain from this that for a suitable choice of a,

$$|y_1|U/3 \le \big|x_1(y) - \mathfrak{g}(x_2(y))\big| \le |y_1|5U/3 \text{ for } y \in Q_a. \tag{11.7}$$

Equations (11.4) imply the identity

$$\partial_{y_1}\Phi_0(y) \equiv \nabla F_0(\mathbf{x}(y)) \cdot \mathbf{u}(x(y)) = \nabla F_0(\mathbf{x}(y)) \cdot \mathbf{U}(x(y)) \text{ for } \Phi_0(y) = 0.$$

Combining this result with (11.3) and (11.7), we obtain the estimates

$$|y_1|N^- U/3 \le |\partial_{y_1}\Phi_0(y)| \le |y_1|N^+ U5/3,$$

which along with the identity

$$\partial_{y_1}\Phi = -\partial_{y_1}\Phi_0(\partial_{y_3}\Phi_0)^{-1}$$

yield the inequalities

$$\begin{gathered} -c < \partial_{y_1}\Phi(y_1, y_2) \le cy_1 \text{ for } -a < y_1 < 0, \\ cy_1 < \partial_{y_1}\Phi(y_1, y_2) \le c \text{ for } 0 < y_1 < a, \\ |\partial_{y_2}\Phi(y_1, y_2) \le c, 0 \le \Phi(y_1, y_2) \le cy_1^2. \end{gathered} \tag{11.8}$$

It is clear that for sufficiently small a, depending only on $\mathbf{U}$ and Ω, the functions $\Phi^\pm(y_2) = \Phi(\pm a, y_2)$ admit the estimates $ca^2 \le \Phi^\pm(y_2) < a$. Set

$$Q_{in} = \{Y \in [-a, a] \times [0, a] : \ 0 < y_3 < \Phi^-(y_2)\},$$
$$Q_{out} = \{Y :\in [-a, a] \times [0, a] : \ 0 < y_3 < \Phi^+(y_2)\}.$$

It follows from (11.8) that for every $Y \in Q_{in}$ ($Y \in Q_{out}$) the equation $y_3 = \Phi(y_1, y_2)$ has a unique solution $a^-(Y) < 0$, ($a^+(Y) > 0$). We adopt the convection that $a^\pm(Y) = \pm a$ for $y_3 > \Phi^\pm(y_2)$. It remains to note that, by virtue of (11.8), the functions $a^\pm$ meet all requirements of Lemma 9.2.

Proof of Lemma 9.3. The proof is similar to the proof of Lemma 9.2. Choose the local Cartesian coordinates (x_1, x_2, x_3) centered at P such that in new coordinates $\mathbf{n} = \mathbf{e}_3$. By the smoothness of $\partial\Omega$, there is a neighborhood $\mathcal{O} = [-k, k]^2 \times [-t, t]$ such that the manifold $\partial\Omega \cap \mathcal{O}$ is defined by the equation

$$x_3 = F(x_1, x_2), \quad F(0,0) = 0, \quad |\nabla F(x_1, x_2)| \le K(|x_1| + |x_2|).$$

The constants k, t and K depend only on Ω. Let us consider the initial value problem

$$\partial_{y_3}\mathbf{x} = \mathbf{u}(\mathbf{x}(y)) \text{ in } Q_a, \quad \mathbf{x}\Big|_{y_3=0} = (y_1, y_2, F(y_1, y_2)). \tag{11.9}$$

Without any loss of generality we can assume that $0 < b < k < 1$. For any such b, problem (11.9) has a unique solution of class $C^1(Q_b)$. Next, note that for $y_3 = 0$ we have

$$|\mathbf{x}(y)| \le (K+1)|y|, \quad |\mathbf{u}(\mathbf{x}(y)) - \mathbf{u}(0)| \le M(K+1)|y|. \tag{11.10}$$

Denote by $\mathfrak{F}(y) = D_y\mathbf{x}(y)$. The calculations show that

$$\mathfrak{F}_0 := \mathfrak{F}(y)\Big|_{y_3=0} = \begin{pmatrix} 1 & 0 & u_1 \\ 0 & 1 & u_2 \\ 0 & 0 & u_3 \end{pmatrix}, \quad \mathfrak{F}(0) = \begin{pmatrix} 1 & 0 & u_1(P) \\ 0 & 1 & u_2(P) \\ 0 & 0 & U_n \end{pmatrix},$$

which along with (11.10) implies

$$\|\mathfrak{F}(0)^{\pm 1}\| \le C/3, \quad \|\mathfrak{F}_0(y) - \mathfrak{F}(0)\| \le cb. \tag{11.11}$$

Next, differentiation of (11.9) with respect to y leads to the equation

$$\partial_{y_1}\mathfrak{F} = D_y\mathbf{u}(\mathbf{x})\mathfrak{F}, \quad \mathfrak{F}\Big|_{y_3=0} = \mathfrak{F}_0.$$

Arguing as in the proof of Lemma 9.2 we obtain $\|\mathfrak{F} - \mathfrak{F}_0\| \le c(M)\|\mathfrak{F}_0\|b$. Combining this result with (11.11) we finally arrive at $\|\mathfrak{F}(y) - \mathfrak{F}(0)\| \le cb$. From this and the implicit function theorem we conclude that there is $b > 0$, depending only on M and Ω, such that the mapping $x = \mathbf{x}(y)$ takes diffeomorphically the cube Q_b onto some neighborhood of the point P, and satisfies inequalities (9.9). Inclusions (9.10) easily follows from (9.9).

Appendix B. Proof of Lemma 9.4

Throughout of the section the notation c, C stands for various constants depending only on the domain $\mathcal{P}_a$ and exponents s, r. Furthermore, for any $y = (y_1, y_2, y_3)$ and $z = (z_1, z_2, z_3)$ we shall write Y and Z for (y_2, y_3) and (z_2, z_3), respectively.

The existence and uniqueness of solutions to problem (9.11) is obvious. Multiplying both sides of equation (9.11) by $|\varphi|^{r-1}\varphi$ and integrating the result over $\mathcal{P}_a$ we obtain the inequality

$$\|\varphi\|_{L^r(\mathcal{P}_a)} \le \sigma^{-1}\|f\|_{L^r(\mathcal{P}_a)} \text{ for } r < \infty. \tag{12.1}$$

Letting $r \to \infty$ we conclude that (12.1) holds true for $r = \infty$. Let us turn to the proof of inequality (9.12) and begin with the case $s = 1$. First we derive the estimates for $\partial_{y_k}\varphi$, $k = 2, 3$. For every $y \in \mathbb{R}^3$, we denote by $Y = (y_2, y_3)$. The function $\partial_{y_k}\varphi$ has the representation $\partial_{y_k}\varphi = \varphi' + \varphi''$, where

$$\varphi'(y) = -e^{\sigma(y_1 - a^-(Y))}\partial_{y_k}a^-(Y)f(a^-(Y), Y) \text{ for } k = 2, 3,$$

$$\varphi'(y) = -e^{\sigma(y_1 - a^-(Y))}(Y)f(a^-(Y), Y) \text{ for } k = 1,$$

and φ'' is a solution to boundary value problem

$$\partial_{y_1}\varphi'' + \sigma\varphi'' = \partial_{y_k} f \text{ in } \mathcal{P}_a, \quad \varphi''(y) = 0 \text{ for } y_1 = a^-(Y).$$

It follows from (12.1) that $\|\varphi''\|_{L^r(\mathcal{P}_a)} \le \sigma^{-1}\|f\|_{H^{1,r}(\mathcal{P}_a)}$. On the other hand, inequalities (9.2) yield the estimate

$$\int\limits_{a^-(Y)}^{a^+(Y)} |\varphi'(y_1, Y)|^r \, dy_1 \le c(r)\sigma^{-1}\|f\|^r_{L^\infty(\mathcal{P}_a)} y_3^{-r/2}(1 - e^{r\sigma(a^-(Y) - a^+(Y))}),$$

Since $0 \le a^+ - a^- \le c y_3^{1/2}$ we conclude from this that

$$\|\varphi'\|^r_{L^r(\mathcal{P}_a)} \le c\sigma^{-1}\|f\|^r_{L^\infty(\mathcal{P}_a)} \int\limits_0^a y_3^{-r/2}(1 - e^{-c\sigma\sqrt{y_3}}) dy_3. \tag{12.2}$$

We have

$$\sigma^{-1}\int\limits_0^a y_3^{-r/2}(1-e^{-c\sigma\sqrt{y_3}})dy_3 = \sigma^{r-3}\int\limits_0^{a\sigma^2} t^{-r/2}(1-e^{-c\sqrt{t}})\, dt \le \begin{cases} c\sigma^{r-3} & \text{for } \quad 2 < r < 3 \\ c\sigma^{-1}\log\sigma & \text{for } \quad r = 2 \\ c\sigma^{-1} & \text{for } \quad 1 < r < 2 \end{cases}$$

Thus, we get

$$\|\varphi'\|_{L^r(\mathcal{P}_a)} \le c\|f\|_{L^\infty(\mathcal{P}_a)} \begin{cases} c\sigma^{-1+\alpha} & \text{for } \quad r \in (1,2)\cup(2,3) \\ c\sigma^{-1+\alpha}\log\sigma & \text{for } \quad r = 2 \end{cases}$$

where $\alpha = \max\{0, 1 - 1/r, 2 - 3/r\}$. Combining the estimates for φ' and φ'' we obtain (9.12).

The proof of inequality (9.12) for $0 < s < 1$ is more complicated. By virtue of (12.1), it suffices to estimate the semi-norm $|\varphi|_{s,r,\mathcal{P}_a}$. Since the expression (1.2) for this semi-norm is invariant with respect to the permutation $(Y, Z) \to (Z, Y)$, we have

$$|\varphi|_{s,r,\mathcal{P}_a} \le (2I)^{1/r}, \quad I = \int_{D_a} |\varphi(z) - \varphi(y)|^r |z - y|^{-3-rs}\, dxdy, \tag{12.3}$$

where $D_a = \{(y, z) \in (\mathcal{P}_a)^2 : a^-(Z) \le a^-(Y)\}$. It is easy to see that

$$\varphi(z) - \varphi(y) = \varphi(z_1, Z) - \varphi(y_1, Z) + \int\limits_{a^-(Z)}^{a^-(Y)} e^{\sigma(x_1 - y_1)} f(x_1, Z)\, dx_1 +$$

$$\int\limits_{a^-(Y)}^{y_1} e^{\sigma(x_1 - y_1)}(f(x_1, Z) - f(x_1, Y))\, dx_1 = I_1 + I_2 + I_3. \tag{12.4}$$

Hence our task is to estimate the quantities

$$J_k = \int_{D_a} |I_k|^r |z - y|^{-3-rs}\, dydz, \quad k = 1, 2, 3. \tag{12.5}$$

The evaluation falls naturally into three steps and it is based on the following proposition

Proposition B.1. *If $r, s > 0$ and $i \neq j \neq k$, $i \neq k$, then*

$$\int\limits_{[-a,a]^2} |z-y|^{-3+rs}\, dy_i dy_j \leq c(r,s)|z_k - y_k|^{-1-rs}.$$

Proof. Note that the left-hand side of the desired equality is equal to

$$|z_k - y_k|^{-1-rs} \int\limits_{[-a,a]^2} \big(1 + \frac{|z_i - y_i|^2 + |z_j - y_j|^2}{|z_k - y_k|^2}\big)^{(-2-rs)/2} \frac{dy_i dy_j}{|z_k - y_k|^2} \leq$$

$$c(r,s)|z_k - y_k|^{-1-rs} \int\limits_{\mathbb{R}^2} (1 + |y_i|^2 + |y_j|^2)^{-(3+rs)/2}\, dy_i dy_j,$$

and the proposition follows. □

Step 1. We begin with the observation that, by virtue of the extension principle, the right- hand side f has an extension over $\mathbb{R}^3$, which vanishes outside the cube Q_{3a} and satisfies the inequalities

$$\|f\|_{H^{s,r}(\mathbb{R}^3)} \leq c(a,r,s)\|f\|_{H^{s,r}(Q_a)}, \quad \|f\|_{L^\infty(\mathbb{R}^3)} \leq \|f\|_{L^\infty(Q_a)}. \tag{12.6}$$

Next recall that $a^-(Z) \leq y_1, z_1 \leq a$ for all $(y,z) \in D_a$. From this and Proposition B.1 we obtain

$$J_1 \leq \int\limits_{[-a,a]^2} \Big\{ \int\limits_{[a^-(Z),a]^2} |\varphi(z_1,Z) - \varphi(y_1,Z)|^r \Big\{ \int\limits_{[-a,a]^3} |z-y|^{-3-rs} dy_2 dy_3 dz_2 \Big\} dy_1 dz_1 \Big\} dZ$$
$$\leq \int\limits_{[-a,a]^2} \Big\{ \int\limits_{[a^-(Z),a]^2} |\varphi(z_1,Z) - \varphi(y_1,Z)|^r |y_1 - z_1|^{-1-rs}\, dy_1 dz_1 \Big\} dZ.$$

Since the right- hand side of this inequality is invariant with respect to the permutation $(y_1, z_1) \to (z_1, y_1)$, we have

$$J_1 \leq c(r,s) \int\limits_{[-a,a]^2} \Big\{ \int\limits_{D(Z)} |\varphi(z_1,Z) - \varphi(y_1,Z)|^r |y_1 - z_1|^{-1-rs}\, dy_1 dz_1 \Big\} dZ, \tag{12.7}$$

where $D(Z) = \{(y_1, z_1) : a^-(Z) \leq z_1 \leq y_1 \leq a\}$. Next note that for all $(y_1, z_1) \in D(Z)$,

$$\varphi(y_1, Z) = \int\limits_{a^-(Z)}^{y_1} e^{\sigma(t-y_1)} f(t,Z)\, dt = \int\limits_{a^-(Z)-\xi}^{z_1} e^{\sigma(t-z_1)} f(t+\xi, Z)\, dt,$$

where $\xi = y_1 - z_1 > 0$. Thus, we get

$$\varphi(y_1,Z) - \varphi(z_1,Z) = \int\limits_{a^-(Z)}^{z_1} e^{\sigma(t-z_1)} (f(t+\xi,Z) - f(t,Z))\, dt + \tag{12.8}$$
$$\int\limits_{a^-(Z)-\xi}^{a^-(Z)} e^{\sigma(t-z_1)} f(t+\xi, Z)\, dt \,:=\, I_{11} + I_{12}.$$

Since f is extended over $\mathbb{R}^3$, we have the estimate

$$\int_{D(Z)} |I_{11}|^r |y_1 - z_1|^{-1-rs}\, dy_1 dz_1 \le \int_{a^-(Z)}^{a} \int_0^{2a} |M(z_1,\xi,Z)|^r\, dz_1 d\xi,$$

$$\text{where } M(z_1,\xi,Z) = \xi^{-s-1/r} \int_{a^-(Z)}^{z_1} e^{\sigma(t-z_1)}(f(t+\xi,Z) - f(t,Z))\, dt.dz_1 d\xi$$

It is easy to see that M satisfies the equation and boundary condition

$$\partial_{z_1} M + \sigma M = K \text{ for } z_1 \in (a^-(Z), a), \quad M = 0 \text{ for } z_1 = a^-(Z),$$

where $K(z_1,\xi,Z) = \xi^{-s-1/r}(f(z_1+\xi,Z) - f(z_1,Z))$. Multiplying both sides of this equation by $|M|^{r-2}M$ and integrating the result over the interval $(a^-(Z), a)$ we arrive at the inequality

$$\sigma \int_{a^-(Z)}^{a} |M|^r\, dz_1 \le \int_{a^-(Z)}^{a} |M|^{r-1}|K|\, dz_1 \le \Big(\int_{a^-(Z)}^{a} |M|^r\, dz_1\Big)^{1-1/r} \Big(\int_{a^-(Z)}^{a} |K|^r\, dz_1\Big)^{1/r},$$

which gives

$$\int_{a^-(Z)}^{a} |M(z_1,\xi,Z)|^r\, dz_1 \le \sigma^{-r}\xi^{-1-rs} \int_{a^-(Z)}^{a} |f(z_1+\xi,Z) - f(z_1,Z)|^r\, dz_1.$$

Recalling that f is extended over $\mathbb{R}^3$ and vanishes outside the cube Q_{3a} we obtain the following estimate for the quantity I_{11} in the right-hand side of (12.8),

$$(12.9) \qquad \int_{[-a,a]^2} \int_{D(Z)} |I_{11}|^r |y_1 - z_1|^{-1-rs}\, dy_1 dz_1 dZ \le$$

$$\sigma^{-r} \int_{[-a,a]^2} \int_{a^-(Z)}^{a} \int_0^{2a} \xi^{-1-rs} |f(z_1+\xi,Z) - f(z_1,Z)|^r\, dz_1 d\xi dZ \le$$

$$\sigma^{-r} \int_{\mathbb{R}^4} |f(y_1,Z) - f(z_1,Z)|^r |y_1 - z_1|^{-1-rs} dy_1 dz_1 dZ \le c\sigma^{-r} \|f\|^r_{L^r(\mathbb{R}^2; H^{s,r}(\mathbb{R}))} \le c\sigma^{-r}\|f\|^r_{H^{s,r}(\mathbb{R}^3)}.$$

In order to estimate I_{12} note that

$$|I_{12}| = \Big| \int_{a^-(Z)-\xi}^{a^-(Z)} e^{\sigma(t-z_1)} f(t+\xi,Z)\, dt \Big| \le \|f\|_{L^\infty(Q_{2a})} e^{\sigma(a^-(Z)-z_1)}\sigma^{-1}(1 - e^{-\sigma\xi}).$$

which gives

$$\int_{D(Z)} |I_{12}|^r |y_1 - z_1|^{-1-rs}\, dy_1 dz_1 \le c\sigma^{-r}\|f\|^r_{L^\infty(Q_{2a})} \int_{a^-(Z)}^{a} e^{r\sigma(a^-(Z)-z_1)}\, dz_1 \int_0^{2a} \xi^{-1-rs}(1 - e^{-\sigma\xi})^r d\xi \le$$

$$c\sigma^{-1-r+rs}\|f\|^r_{L^\infty(Q_{2a})} \int_0^{\infty} \xi^{-1-rs}(1 - e^{-\xi})^r d\xi \le c\sigma^{-1-r+rs}\|f\|^r_{L^\infty(Q_{2a})}.$$

¿From this we conclude that

$$\int\limits_{[-a,a]^2}\int\limits_{D(Z)} |I_{12}|^r|y_1-z_1|^{-1-rs}\,dy_1dz_1dZ \le c\|f\|^r_{L^\infty(Q_{2a})}\sigma^{-1-r+rs}$$

Substituting this inequality together with (12.9) in (12.7), and recalling inequalities (12.6) we finally obtain the estimate

$$J_1^{1/r} \le c\big(\sigma^{-1}\|f\|_{H^{r,s}(Q_a)} + \sigma^{-1+(s-1/r)}\|f\|_{L^\infty(Q_a)}\big). \tag{12.10}$$

Step 2. Our next task is to estimate the quantity J_2. It follows from (12.8) that

$$|I_2| \le c\|f\|_{L^\infty(\mathcal{P}_a)}\sigma^{-1}e^{\sigma(a^-(Y)-y_1)}\big(1-e^{\sigma(a^-(Z)-a^-(Y))}\big).$$

Since $0 \le a^-(Y)-a^-(Z) \le c|z_2-y_2| + c|\sqrt{z_3}-\sqrt{y_3}|$, we have

$$J_2 = \int_{D_a}|I_2|^r|z-y|^{-3-rs} \le c\sigma^{-r}\|f\|^r_{L^\infty(Q_a)}(J_{23}+J_{22}),\ \text{where} \tag{12.11}$$

$$J_{23} = \int_{D_a} e^{r\sigma(a^-(Y)-y_1)}\big(1-e^{-c\sigma|\sqrt{y_3}-\sqrt{z_3}|}\big)^r|z-y|^{-3-rs}dydz,$$

$$J_{22} = \int_{D_a} e^{r\sigma(a^-(Y)-y_1)}\big(1-e^{-c\sigma|y_2-z_2|}\big)^r|z-y|^{-3-rs}dydz.$$

It follows from the obvious inclusion

$$D_a \subset \{(y,z):\ y_2,z_1,z_2\in(-a,a),\ y_3,z_3\in(0,a),\ a^-(Y)<y_1<a^+(Y)\}$$

that

$$J_{23} \le \int\limits_{[0,a]^2}\Big\{\int\limits_a^a\Big[\int\limits_{a^-(Y)}^{a^+(Y)}\Big(\int\limits_{[0,a]^2}|z-y|^{-3-rs}\,dz_1dz_2\Big)e^{r\sigma(a^-(Y)-y_1)}\,dy_1\Big]\,dy_2\Big\}\big(1-e^{-c\sigma|\sqrt{y_3}-\sqrt{z_3}|}\big)^r\,dy_3dz_3$$

Next, Proposition B.1 along with the inequality $|a^-(Y)-a^+(Y)| \le c\sqrt{y_3}$ yields the estimate

$$\int\limits_{a^-(Y)}^{a^+(Y)}\Big(\int\limits_{[-a,a]^3}|z-y|^{-3-rs}dz_1dz_2\Big)e^{r\sigma(a^-(Y)-y_1)}\,dy_1 \le c|y_3-z_3|^{-1-rs}\int\limits_{a^-(Y)}^{a^+(Y)}e^{r\sigma(a^-(Y)-y_1)}\,dy_1 \le$$

$$c\sigma^{-1}|y_3-z_3|^{-1-rs}\big(1-e^{r\sigma(a^-(Y)-a^+(Y))}\big) \le c\sigma^{-1}|y_3-z_3|^{-1-rs}\big(1-e^{-c\sigma\sqrt{y_3}}\big).$$

which leads to the inequality

(12.12)

$$J_{23} \le c\sigma^{-1}\int\limits_0^a S(z_3)\,dz_3, \quad S(z_3) = \int\limits_0^a(1-e^{-c\sigma\sqrt{y_3}})\big(1-e^{-c\sigma|\sqrt{y_3}-\sqrt{z_3}|}\big)^r|y_3-z_3|^{-1-rs}\,dy_3.$$

The change of the variable $t = \sqrt{y_3/z_3}-1$ gives the inequality

$$S \le cz_3^{-rs}\int\limits_{-1}^{\infty}(1-e^{-c\mu(t+1)})(1-e^{-c\mu|t|})^r|t(t+2)|^{-1-rs}(t+1)\,dt \le$$

$$cz_3^{-rs}\int\limits_{-1}^{1}(1-e^{-c\mu|t|-c\mu})(1-e^{-c\mu|t|})^r|t|^{-1-rs}\,dt + cz_3^{-rs}\int\limits_1^{\infty}(1-e^{-c\mu t})^{r+1}t^{-1-2rs}\,dt,$$

where $\mu = \sigma\sqrt{z_3}$. The second change of the variable $\tau = \mu t$ along with the identity $z_3^{-rs} = \sigma^{2rs}\mu^{-2rs}$ yields the estimate

$$S(z_3) \le c\sigma^{2rs}(S_1(\mu) + S_\infty(\mu)),$$

where

$$S_1 = \mu^{-rs}\int_0^\mu (1-e^{-c\tau-c\mu})(1-e^{-c\tau})^r\tau^{-1-rs}\,d\tau, \quad S_\infty = \int_\mu^\infty (1-e^{-c\tau})^{r+1}\tau^{-1-2rs}\,d\tau\,.$$

Note that the inequality $0 < s < 1$ guarantees the convergence of these integrals. Substituting this estimate into (12.12) we finally obtain

$$J_{23} \le c\sigma^{2rs-1}\int_0^a (S_1(\mu) + S_\infty(\mu))\,dz_3 = c\sigma^{2rs-3}\int_0^{\sigma\sqrt{a}} (S_1(\mu) + S_\infty(\mu))\mu\,d\mu \tag{12.13}$$

Since $(1 - e^{-c\tau-c\mu})(1 - e^{-c\tau})^r \le c\tau^{r+1} + c\mu\tau^r$ and $(1 - e^{-c\tau})^{r+1} \le c\tau^{r+1}$, we have the estimates

$$\mu S_1(\mu) \le c\mu^{r-2rs+2}, \quad \mu S_\infty(\mu) \le c\mu + c\mu^{r-2rs+2} \text{ for all } \mu \in (0,1).$$

¿From this and the inequality $r - 2rs + 2 = -1 + r(1-\kappa) > -1$ we conclude that the integrals in (12.13) converge at 0, and are finite for each finite σ. On the other hand, we have for $\mu \ge 1$,

$$\mu S_1(\mu) \le c\mu^{1-rs}, \quad \mu S_\infty(\mu) \le c\mu^{1-2rs}.$$

Hence for all $\sigma > 1$

$$J_{23} \le c\sigma^{2rs-3}\int_{\sqrt{a}}^{\sigma\sqrt{a}} (\mu^{1-rs}+\mu^{1-2rs})\,d\mu + c\sigma^{2rs-3} \le c\begin{cases} \sigma^{2rs-3} + \sigma^{rs-1} & \text{for} \quad sr \ne 1,2 \\ (\sigma^{2rs-3} + \sigma^{rs-1})(1+\log\sigma) & \text{for} \quad sr = 1,2 \end{cases}$$

Since $(2rs-3), (rs-1) \le r\alpha$, we conclude from this that for all $\sigma > 1$,

$$J_{23}^{1/r} \le c\sigma^\alpha \text{ when } sr \ne 1,2, \text{ and } J_{23}^{1/r} \le c\sigma^\alpha(1+\log\sigma)^{1/r} \text{ when } sr = 1,2, \tag{12.14}$$

Let us estimate the quantity J_{22}. We have

$$J_{22} \le \int_{[-a,a]^2}\int_{[0,a]}\Big\{\int_{[-a,a]^2\times[0,a]} e^{\sigma r(a^-(Y)-y_1)}|z-y|^{-3-rs}dz_1dy_1dz_3\Big\}(1-e^{-c\sigma|y_2-z_2|})^r dy_3dy_2dz_2.$$

Since

$$\int_{[-a,a]\times[0,a]} |z-y|^{-3-rs}dz_1dz_3 \le c|y_2 - z_2|^{-1-rs},$$

we have

$$\int_{[-a,a]^2\times[0,a]} e^{\sigma r(a^-(Y)-y_1)}|z-y|^{-3-rs}dz_1dy_1dz_3 \le c\sigma^{-1}|y_2 - z_2|^{-1-rs},$$

which yields

$$J_{22} \le c\sigma^{-1}\int_{-\infty}^{\infty} (1-e^{-c\sigma|y_2-z_2|})^r|y_2-z_2|^{rs-1}dy_2 = c\sigma^{rs-1}\int_{-\infty}^{\infty} (1-e^{-c|t|})^r|t|^{-1-rs}dt \le c\sigma^{rs-1}.$$

Since for all $\sigma \geq 1$, $\sigma^{rs-1} \leq c\sigma^{r\alpha}$, we conclude from this and inequalities (12.14), (12.11) that
(12.15)

$$J_2^{1/r} \leq c\|f\|_{L^\infty(\mathcal{P}_a)}\sigma^\alpha \text{ for } sr \neq 1,2, \;\; J_2^{1/r} \leq c\|f\|_{L^\infty(\mathcal{P}_a)}\sigma^\alpha(1+\log\sigma)^{1/r} \text{ for } sr = 1,2.$$

Step3. We begin with the observation that the function $I_3(y_1, Y, Z)$ defined by relation (12.4) satisfies the equation and boundary condition

$$\partial_{y_1} I_3 + \sigma I_3 = K_3 \text{ for } a^-(Y) < y_1 < a, \quad I_3(a^-(Y), Y, Z) = 0,$$

where $K_3(y_1, Y, Z) = f(y_1, Z) - f(y_1, Y)$. Multiplying both sides of this equation by $|I_3|^{r-2}I_3$ and integrating the result over the interval $(a^-(Y), a)$ we arrive at the inequality

$$\sigma \int\limits_{a^-(Y)}^{a} |I_3|^r \, dy_1 \leq \int\limits_{a^-(Y)}^{a} |I_3|^{r-1}|K_3| \, dy_1 \leq \Big(\int\limits_{a^-(Y)}^{a} |I_3|^r \, dy_1\Big)^{1-1/r}\Big(\int\limits_{a^-(Y)}^{a} |K_3|^r \, dy_1\Big)^{1/r},$$

which leads to the estimate

$$\int\limits_{a^-(Y)}^{a} |I_3|^r \, dy_1 \leq \sigma^{-r} \int\limits_{[-a,a]} |f(y_1, Z) - f(y_1, Y)|^r \, dy_1.$$

Since $a^-(Y) \leq y_1$ for all $(y, z) \in D_a$, we conclude from this and the inequality

$$\int\limits_{[-a,a]} |z-y|^{-3-rs}dz_1 \leq c|Y-Z|^{-2-rs}$$

that

$$J_3 = \int\limits_{D_a} |I_3|^r|z-y|^{-3-rs}dydz \leq \tag{12.16}$$

$$c\sigma^{-r} \int\limits_{[-a,a]^3} |f(y_1, Z) - f(y_1, Y)|^r|Y-Z|^{-2-rs}dy_1dYdZ \leq$$

$$c\sigma^{-r}\|f\|^r_{L^r(-a,a;H^{s,r}([-a,a]^2))} \leq c\sigma^{-r}\|f\|^r_{H^{s,r}(Q_a)}.$$

Combining estimates (12.10), (12.15), and recalling inequalities (12.6) we conclude that under the assumptions of the lemma,

$$\|\varphi\|_{s,r\mathcal{P}_a} \leq \big(\sigma^{-1}\|f\|_{H^{s,r}(\mathcal{P}_a)} + \sigma^{-1+\alpha}\|f\|_{L^\infty(\mathcal{P}_a)}\big) \text{ for } sr \neq 1,2$$

$$\|\varphi\|_{s,r\mathcal{P}_a} \leq \big(\sigma^{-1}\|f\|_{H^{s,r}(\mathcal{P}_a)} + \sigma^{-1+\alpha}(1+\log\sigma)^{1/r}\|f\|_{L^\infty(\mathcal{P}_a)}\big) \text{ for } sr = 1,2$$

which completes the proof. □

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